Barycentric Coordinates

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Contents

Chapter 1

Barycentric Coordinates

1.1 Introduction

Barycentric coordinates were first introduced by August Ferdinand Möbius (1790 - 1816) in his book *The barycentric calculus*, published in 1827 (Fauvel, 1993). He began with the idea of a weightless rod with weights attached at two distinct points, and was interested in locating the rod's centre of gravity. In other words, he wanted to find the point at which a fulcrum could be placed to balance the rod (Wildberger, 2010). In his calculations, Möbius also assigned negative weights. Although this may seem counter-intuitive, a negative weight may be considered an object which applies an upward force, such as a balloon. In this case, notice that the centre of gravity is not between the two attached objects.

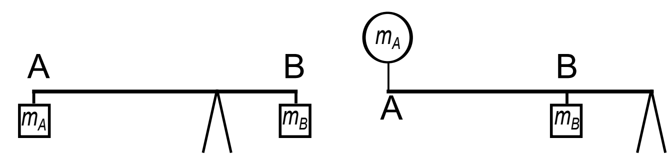


Figure 1.1

Möbius assigned coordinates to the point at which the centre of gravity, or *barycentre*, was located, which reflected the ratio of weights attached to the rod. We write the barycentric coordinates of the point at which the fulcrum is located in *Figure 1* as $(m_a : m_b)$. The particular location of the fulcrum will be examined further in Section 1.3.

Möbius then extended this idea to a system of three weighted points, forming a triangle (Fauvel, 1993). Suppose the weights m_A, m_B and m_C are placed at the corresponding vertices of triangle ABC. Then the barycentric coordinates of the point P, the location of its centre of gravity, are represented by the ratio of weights $(m_A : m_B : m_C)$.

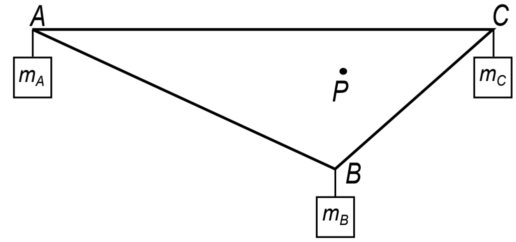


Figure 1.2

If the weights in *Figure 1.2* are replaced with weights that have each been increased or decreased by a common factor, the ratio of weights, and centre of gravity will remain the same. Therefore, a specific point P has many sets of barycentric coordinates. For example, suppose the weights in *Figure 1.2* are assigned the following masses: $m_A = 2$ g, $m_B = 3g$, $m_C = 4$ g. The coordinates assigned to P in this case are $(2 : 3 : 4)$. If these weights are then replaced such that $m_A = 4$ g, $m_B = 6$ g, and $m_C = 8$ g, the coordinates assigned to point P are $(4 : 6 : 8)$. The location of P, however, has not changed. Therefore, the barycentric coordinates of this point P may be denoted by $(\lambda 2 : \lambda 3 : \lambda 4)$ where $\lambda \in \mathbb{R}$ and are said to be *homogeneous*.

The coordinates $(u : v : w)$ are often *normalized* such that the values sum to 1, that is $u + v + w = 1$ in order to provide easier calculations (Abel, 2007). To normalize coordinates $(u : v : w)$, simply multiply each value by $\frac{1}{u+v+w}$. Therefore, the normalized coordinates of $P(u : v : w)$ are:

$$\frac{1}{u+v+w}(u : v : w) = \left(\frac{u}{u+v+w}, \frac{u}{u+v+w}, \frac{u}{u+v+w}\right).$$

Normalized coordinates are often represented with commas instead of colons. Note that in the case where $u + v + w = 0$, such scaling is not possible.

1.2 Advantages

As we progress with the presentation of topics surrounding barycentric coordinates, some of the advantages of this system of coordinates will become apparent. They include the simple expressions for lines, the roles that the vertices and edges of a triangle play, and the simple forms for common points of a triangle (Schindler & Chen, 2012).

Although the use of barycentric coordinates extends to problems of complexity beyond the scope of the content presented here, it is worth mentioning their "extreme helpfulness for interpolating discrete scalar fields, vector fields, or arbitrary multidimensional fields over irregular tessellations" (Warren, 2006).

1.3 Locating the barycentre

Referring back to the weightless rods in *Figure 1.1* with weights m_A and m_B attached at points A and B respectively, let us determine the location of the fulcrum. If d represents the distance between A and B, the centre of gravity is a distance of $\frac{m_B}{m_A+m_B}d$ from A and $\frac{m_A}{m_A+m_B}d$ from B.

For example, consider the rod depicted below with attached weights of mass 3 kg at A and 5 kg at B. If the distance between A and B is again represented by d, then the location of P is $\frac{5}{8}d$ from A and $\frac{3}{8}d$ from B.

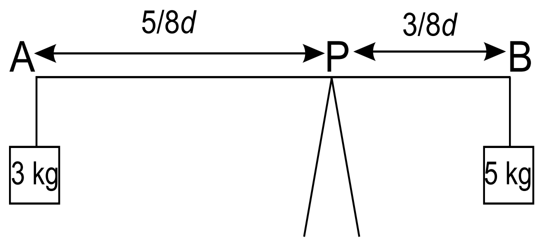


Figure 1.3

In the case where the fulcrum is not located between the weights, if $m_A < 0$ and $m_A + m_B > 0$, then $\frac{m_A}{m_A+m_B}d < 0$. Although the distance is positive, the negative value indicates the position of B is between the fulcrum and A. This will be investigated further near the end of this section.

Extending this now to a triangle, such as the one depicted in *Figure 1.2*, the location of P may not be on one of the edges of the triangle. In this case, determining the location of P involves a few more steps. Consider the triangle below.

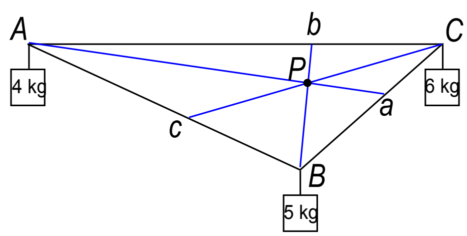


Figure 1.4

Applying the calculations done with the rod in *Figure 1.3*, the locations of a, b and c can be determined. Let the distances between vertices be represented by d_{AB}, d_{BC}, and d_{CA}. For the triangle in *Figure 1.4*, the point a is located $\frac{5}{11}d_{BC}$ from C and $\frac{5}{11}d_{BC}$ from B. Point b is $\frac{6}{10}d_{CA}$ from A and $\frac{4}{10}d_{CA}$ from C, and point c is $\frac{5}{9}d_{AB}$ from A and $\frac{4}{9}d_{AB}$ from B. P is located at the point of intersection of the lines Aa, Bb and Cc. These are the *cevians* of the triangle and will be adressed in Ceva's Theorem. The equation of a line will be also be investigated further in Chapter 2.

It should now be clear that the vertices of the reference triangle ABC define the barycentric coordinate system, and so the coordinates of all other points P in the plane are relative to the locations of A, B and C. While the steps outlined above may be somewhat tedious to determine the position of a single point, there are some points which are easy to locate. In the following, $\triangle ABC$ will serve as the reference triangle.

Vertices:

Consider the *normalized* points with barycentric coordinates $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. They each represent an attached weight at a single vertex, and are located at that respective vertex. More precisely, $(1, 0, 0)$ is located at A, $(0, 1, 0)$ is located at B, and $(0, 0, 1)$ is located at C.

Midpoints:

Now consider the point with coordinates $(1 : 1 : 0)$. Since the weights placed at A and B are equal, and no weight is placed at C, the location of the barycentre is on the line passing through, and equidistant to, A and B. In other words, it is the midpoint of AB. Similarly, $(1 : 0 : 1)$ is the midpoint of AC and $(0 : 1 : 1)$ is the midpoint of BC.

Centroid:

The point $(1 : 1 : 1)$ does not lie on an edge, nor at a vertex of the triangle. This is the barycentre of a triangle with equal weights at each vertex. It is located at the point of concurrency of all three medians of a triangle (Unger, 2010), and is known as the *centroid*. It is always located within the triangle.

The coordinates of a point contain only positive values if and only if the point lies within the triangle of reference. If one of the coordinates is zero, the point is located on one of the edges, if two are zero it lies on a vertex, and if any coordinates are negative, the point is located exterior to the triangle (Lovering, 2008). *Figure 1.5* shows the signs of coordinates at differents locations in reference with an arbitrary triangle.

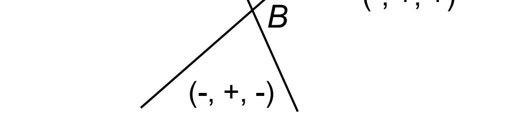


Figure 1.5

1.4 Barycentric coordinates representing area

Normalized barycentric coordinates are sometimes called *areal coordinates*. This is because they are also the ratio of areas into which the triangle has been partitioned using a fourth point P (Lidberg, P). If we consider triangle ABC with coordinates labelled counterclockwise and point P in the same plane, then it can be shown that the coordinates of P are

$$\left(\frac{[PBC]}{[ABC]}, \frac{[PCA]}{[ABC]}, \frac{[PAB]}{[ABC]}\right)$$

where $[ABC]$ represents the area of the triangle with vertices A, B and C (Lovering, 2011).

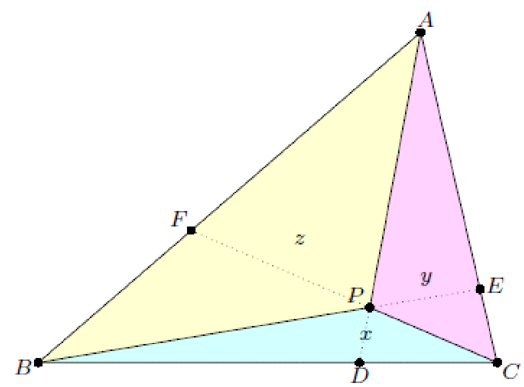


Figure 1.6, presented in Abel, 2007.

The area of a triangle is considered positive if the vertices are labelled counterclockwise. Therefore, $[ABC] > 0$. It is not possible to calculate the area of a triangle based only on its ratio of side lengths, however the areas of the triangles formed with the fourth point P are the product of their determinant and the area of the reference triangle ABC (Chen, 1012). The coordinates of the points are $P(x, y, z)$ where $x+y+z = 0$, $A(1, 0, 0)$, $B(0, 1, 0)$ and $C(0, 0, 1)$. Thus, we have:

$$[PBC] = \begin{vmatrix} x & y & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \cdot [ABC] = x \cdot [ABC],$$
$$[PCA] = \begin{vmatrix} x & y & z \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} \cdot [ABC] = y \cdot [ABC],$$
$$[PAB] = \begin{vmatrix} x & y & z \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} \cdot [ABC] = z \cdot [ABC],$$

which yields:

$$\left(\frac{[PBC]}{[ABC]}, \frac{[PCA]}{[ABC]}, \frac{[PAB]}{[ABC]}\right) = \left(\frac{x \cdot [ABC]}{[ABC]}, \frac{y \cdot [ABC]}{[ABC]}, \frac{z \cdot [ABC]}{[ABC]}\right) = (x, y, z).$$

1.5 A comparison with the Cartesian coordinate system

Consider the triangle ABC in the Cartesian plane with centre of gravity P as depicted below. In this section Cartesian coordinates are represented with round brackets, while barycentric coordinates are represented by square brackets and colons, even for normalized coordinates. This is merely to further help in distinguishing between the two systems.

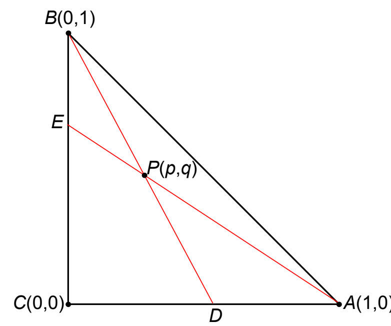


Figure 1.7

The coordinates of the vertices are easy to compare. Placing weights at each vertex individually, the normalized coordinates are $A[1:0:0]$, $B[0:1:0]$ and $C[0:0:1]$. These do not, however, shed much light on the relationship between the two systems of coordinates. To do so, we must investigate the point P.

The slope of the line passing through points A and P is $m = \frac{q-0}{p-1} = \frac{q}{p-1}$. By substituting the coordinates of $A(1,0)$ into the point-slope equation of a line, $y - y_1 = m(x - x_1)$, we can determine the equation of the line passing through A and P to be $y = \frac{q}{p-1}(x-1)$. Since point E is on the y-axis, substituting $x = 0$ into the equation yields $y = \frac{q}{1-p}$ and therefore the Cartesian coordinates of E are $(0, \frac{q}{1-p})$.

Since the point E is located where $y = \frac{q}{1-p}$, the distance from C to E is $\frac{q}{1-p}$ and the length of EB is $1 - \frac{q}{1-p} = \frac{1-p-q}{1-p}$. The ratio $CE : EB = \frac{q}{1-p} : \frac{1-p-q}{1-p} = q : 1-p-q$, and with weight $m_A = 0$ at A, the barycentric coordinates of E are $[0 : q : 1-p-q]$.

Using similar steps we can determine the equation of the line passing through points B and P to be $y - 1 = \frac{1-q}{0-p}x$ or $y = \frac{q-1}{p}x + 1$. Since point D is on the x-axis, substituting $y = 0$ into the equation yields $x = \frac{p}{1-q}$ and thus the Cartesian coordinates of D are $(\frac{p}{1-q}, 0)$.

The lengths of CD and DA are therefore $\frac{p}{1-q}$ and $1 - \frac{p}{1-q} = \frac{1-p-q}{1-q}$, respectively. The ratio $CD : DA = \frac{p}{1-q} : \frac{1-p-q}{1-q} = p : 1-p-q$, and with weight $m_B = 0$ at B, the barycentric coordinates of D are $[p : 0 : 1-p-q]$.

Using the equivalent coordinates of points D and E, the relationship can now be established. Since the ratio of the first coordinate to the third coordinate remains consistent on the line BD, and similarly the ratio of the second coordinate to the third coordinate remains consistent on the line AE, the Cartesian coordinates (p, q) are equivalent to barycentric coordinates $[p : q : 1-p-q]$.

Chapter 2

Euclidean Geometry

2.1 The Equation of a Line

The equation of a line in the barycentric coordinate system contains variables of the first degree, and a line passing through two specific points is unique up to scaling. However its similarity with the linear equation in the Cartesian coordinate system does not extend beyond this. An arbitrary triangle replaces the perpendicular lines that are used to structure the Cartesian system, with the edges of the triangle acting as the axes, and the three vertices taking the role of the origin (Chen, 2012). The linear equation in the barycentric coordinate system therefore contains three variables, even though we are still working in a two-dimensional plane. The general equation of a line in the barycentric coordinate system is of the form $\ell x + my + nz = 0$, where ℓ, m, n are constants, and not all zero (Lovering, 2008).

To calculate the values of the constants ℓ, m, n, a general equation can be constructed by applying the determinant method as done in section 1.4. Consider two arbitrary points $P(a_p : b_p : c_p)$ and $Q(a_q : b_q : c_q)$ in the plane. In order to determine the equation of a line passing through them and a third, arbitrary point $(x : y : z)$ we must solve the system:

$$\begin{aligned} (a_p)x + (b_p)y + (c_p)z &= 0 \\ (a_q)x + (b_q)y + (c_q)z &= 0 \end{aligned}$$

Since these points are on a line, the area of the triangle formed by these points must be 0. Therefore, the determinant must equal 0:

$$\begin{vmatrix} x & y & z \\ a_p & b_p & c_p \\ a_q & b_q & c_q \end{vmatrix} = 0.$$

Thus the equation of the line passing through these points can be written as:

$$(b_p c_q - c_p b_q)x + (c_p a_q - a_p c_q)y + (a_p b_q - b_p a_q)z = 0$$

If a line passes through a vertex, such as A, then it passes through $(1, 0, 0)$ so we can substitute these values of x, y, z into the formula to obtain $\ell(1) + m(0) + n(0) = 0$ which tells us that $\ell = 0$. The equation of a line passing through A is therefore $y = kz$ for some $k \in \mathbb{R}$ (Schindler & Chen, 2012). Then, applying the equation above, we can determine the value of k of a line passing through $(1, 0, 0)$ and $(x_q : y_q : z_q)$ to get $-z_q y + y_q z = 0$, which can be arranged to give:

$$y = \frac{y_q}{z_q} z$$

Through similar steps we can determine that the equation of a line passing through $(0, 1, 0)$ has $m = 0$, and a line passing through $(0, 0, 1)$ has $n = 0$. The resulting equations are

$$x = \frac{x_q}{z_q} z \quad \text{and} \quad x = \frac{x_q}{y_q} y$$

respectively. The simplicity of finding the two latter equations is an example of one of the advantages of working with the barycentric coordinate system.

Finally, the case where a line passes through two vertices is to be considered. Substituting $B(0, 1, 0)$ and $C(0, 0, 1)$ into the equation $\ell x + my + nz = 0$ will show that $m, n = 0$ and therefore the equation of the line passing through BC is $x = 0$, which is consistent with the equations for x interms of y and z stated earlier. Similar steps will verify the equation of the line through AB is $z = 0$, and through AC is $y = 0$ (Schindler & Chen, 2012).

2.2 Ceva's Theorem

The Mathematical Database (2012) presents Ceva's Theorem as follows:

Given triangle ABC with points D, E, F on lines passing through BC, AC, AB respectively, if the lines AD, BE and CF intersect at a single point P, then the product of the ratios

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

Note that it is possible for the point of intersection P of the three lines to be inside, or outside, of the triangle (Wikipedia, 2012).

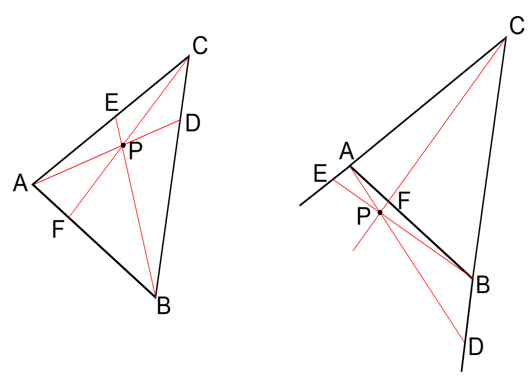


Figure 2.1

Proof: The application of barycentric coordinates provides an elegant proof of Ceva's Theorem. Referencing the proof provided by Schindler & Chen (Schindler & Chen, 2012), since D lies on BC, the normalized barycentric coordinates of point D are $(0, d, 1-d)$ for some $d \in \mathbb{R}$, and $d : 1-d = DC : BD$. If D lies between B and C, as shown with the triangle to the left in *Figure 2.1*, then $0 < d < 1$. However, if D lies on BC and B is located between D and C, $d > 1$. If C lies between D and B, then $d < 0$. Substituting these values into the equation of a line passing through $(0, d, 1-d)$ and $(1, 0, 0)$, we get $(1-d)y - dz = 0$. Rearranging the terms gives us $z = \frac{1-d}{d}y$.

Since the point E lies on AC, the normalized coordinates of E are $(1-e, 0, e)$ for some $e \in \mathbb{R}$, and $e : 1-e = EA : CE$. If E lies between A and C, then $0 < e < 1$. If A lies between E and C, then $e < 0$, and if C lies between A and E, then $e > 1$. The equation passing through $(1-e, 0, e)$ and $(0, 1, 0)$ is $-ex + (1-e)z = 0$. Rearranging the terms yields $x = \frac{1-e}{e}z$.

Finally, since F lies on AC, the point F has the normalized coordinates $F = (f, 1-f, 0)$ for some $f \in \mathbb{R}$, and $f : 1-f = FB : AF$. $0 < f < 1$ when F lies between A and B, $f < 0$ when B lies between A and F, and $f > 1$ when A lies between B and F. Thus the line passing through $(f, 1-f, 0)$ and $(0, 0, 1)$ yields the equation $(1-f)x - fy = 0$, which provides us with $y = \frac{1-f}{f}x$.

Since P lies on all three lines, we can use this system of equations to substitute for x, y, z:

$$\begin{aligned} x &= \frac{1-e}{e}z \\ &= \frac{1-e}{e} \cdot \frac{1-d}{d}y \\ &= \frac{1-e}{e} \cdot \frac{1-d}{d} \cdot \frac{1-f}{f}x \\ 1 &= \frac{1-e}{e} \cdot \frac{1-d}{d} \cdot \frac{1-f}{f} \end{aligned}$$

Since $d : 1-d = DC : BD$, $e : 1-e = EA : CE$ and $f : 1-f = FB : AF$, we can substitute to obtain:

$$\begin{aligned} 1 &= \frac{CE}{EA} \cdot \frac{BD}{DC} \cdot \frac{AF}{FB} \\ &= \frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} \end{aligned}$$

which is Ceva's Theorem.

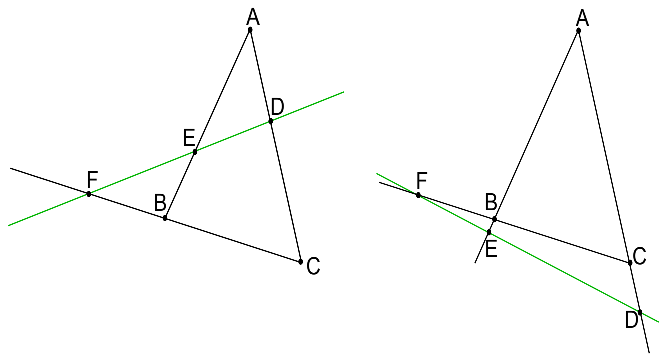
$\square$

2.3 Menelaus' Theorem

Given triangle ABC with points D, E, F on a transversal line crossing AB, BC and AC, respectively, with D, E and F distinct from A, B and C, then

$$\frac{AE}{EB} \cdot \frac{BF}{FC} \cdot \frac{CD}{DA} = -1.$$

As with Ceva's Theorem, there are two possible cases:



Note that the transversal either intersects two edges of the triangle, or lies entirely exterior to the triangle (Wikipedia, 2012).

Proof: The normalized barycentric coordinates of point D are $(d, 0, 1-d)$ where $d \in \mathbb{R}$ and $d : 1-d = CD : DA$. The coordinates of E are $(e, 1-e, 0)$ where $e \in \mathbb{R}$ and $e : 1-e = EB : AE$, and the coordinates of F are $(0, f, 1-f)$ where $f \in \mathbb{R}$ and $f : 1-f = FC : BF$. (Note that in the first case, the value of f is negative since it is located exterior to the triangle, and indicates a negative weight at vertex B. The values of d, e and f are all negative in the second case, indicating that at each point, the weight at one of the weighted vertices on that edge is negative.)

Since D, E and F all lie on the same transversal, we will construct the equation of the line passing through two of those points, and then evaluate using the third point. Lets construct the equation of the line passing through points D and E. Substituting their coordinates into the equation of a line we get:

$$[(0)(0) - (1-e)(1-d)]x + [(1-d)(e) - 0(d)]y + [d(1-e) - e(0)]z = 0$$

which simplifies to

$$-(1-e)(1-d)x + (1-d)(e)y + d(1-e)z = 0$$

We can now substitute the coordinates of $F = (0, f, 1-f)$ into this equation:

$$-(1-e)(1-d)(0) + (1-d)(e)(f) + d(1-e)(1-f) = 0$$

Simplifying and rearranging terms:

$$\begin{aligned}
-(1-d) \cdot e \cdot f &= d \cdot (1-e) \cdot (1-f) \\
-1 &= \frac{d \cdot (1-e) \cdot (1-f)}{(1-d) \cdot e \cdot f} \\
-1 &= \frac{d}{1-d} \cdot \frac{1-e}{e} \cdot \frac{1-f}{f}
\end{aligned}$$

Since $d : 1-d = CD : DA$, $e : 1-e = EB : AE$ and $f : 1-f = FC : BF$, we can substitute to obtain:

$$-1 = \frac{CD}{DA} \cdot \frac{AE}{EB} \cdot \frac{BF}{FC}$$

which is equivalent to Menelaus' Theorem.

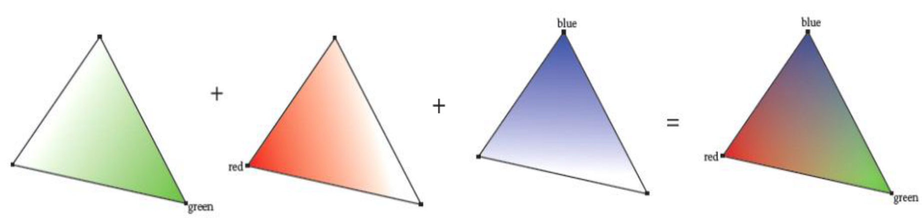
$\square$

Chapter 3

Applications

3.1 Gouraud Shading

Henri Gourard, a computer scientist, first presented Gouraud shading which is used in computer graphics in 1971. It is used to create smooth lighting on polygonal surfaces without having to calculate lighting for each pixel (xtimeline, 2012). Using the shading of a triangle as an example, suppose a pure colour is assigned to each vertex. Using the normalized barycentric coordinates of each point within the triangle, the ratio of colour from each vertex is applied to each pixel. The result is a smooth transition of colour throughout the triangle. The picture below shows the combination of green, red and blue, in a triangle (Lidberg, 2011).



3.2 Solar Chords

In astronomy, the barycentre play an important role. It is the point around which two bodies orbiting each other rotate. In instances where one body of mass is much larger than another, the barycentre is located within the larger mass. When the masses are similar, the barycentre is located at a point between them.

Frederick Bailey, an astronomer, has been plotting the track of the Sun around the barycentre of our solar system. He has discovered that the orbit matches the production of sunspots through changes in acceleration, and has been measuring the correlation between Earth's climate variation and changes in distance between Earth and Sun. As a result, he has been able to present a reliable model for forecasting climate variation, both globally and regionally (Bailey, 2012).

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