Double-Angle Identities

The **double-angle identities** are summarized below.

Double-Angle Identities

$\sin 2\theta = 2\sin\theta\cos\theta \qquad \tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$\cos 2\theta = \cos^2\theta - \sin^2\theta \qquad \cos 2\theta = 2\cos^2\theta - 1 \qquad \cos 2\theta = 1 - 2\sin^2\theta$

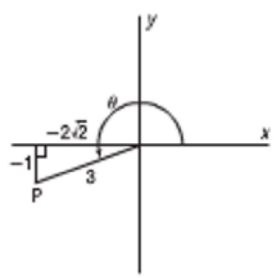
Example 1 Applying the Double-Angle Identities

Given angle θ is in standard position with its terminal arm in Quadrant 3 and $\sin\theta = -\frac{1}{3}$, determine the exact value of each trigonometric ratio.

a) $\sin 2\theta$ **b)** $\tan 2\theta$

SOLUTION

Suppose point $P(x, y)$ lies on the terminal arm of angle θ.
Since the terminal arm of θ lies in Quadrant 3, $\cos\theta$ is negative.
From $\sin\theta = -\frac{1}{3}$, set $y = -1$ and $r = 3$
Sketch a diagram.



Use mental math and the Pythagorean Theorem to determine the x-coordinate of P, which is $-\sqrt{8}$, or $-2\sqrt{2}$.

So, $\cos\theta = -\dfrac{2\sqrt{2}}{3}$ and $\tan\theta = \dfrac{1}{2\sqrt{2}}$

a) $\sin 2\theta$

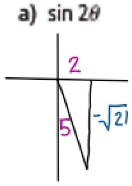
Substitute $\sin\theta = -\frac{1}{3}$ and $\cos\theta = -\frac{2\sqrt{2}}{3}$ in:

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta \\ &= 2\left(-\frac{1}{3}\right)\left(-\frac{2\sqrt{2}}{3}\right) \\ &= \frac{4\sqrt{2}}{9}\end{aligned}$$

Check Your Understanding

1. Given angle θ is in standard position with its terminal arm in Quadrant 4 and $\cos\theta = \frac{2}{5}$ determine the exact value of each trigonometric ratio.

a) $\sin 2\theta$ **b)** $\cos 2\theta$



$5^2 - 2^2$
$= 25 - 4$
$= 21$

a) $\sin 2\theta = 2\sin\theta\cos\theta$
$= 2\cdot\frac{-\sqrt{21}}{5}\cdot\frac{2}{5}$
$= \frac{-4\sqrt{21}}{25}$

b) $\cos 2\theta = 2\cos^2\theta - 1$
$= 2\cdot\left(\frac{2}{5}\right)^2 - 1$
$= 2\cdot\frac{4}{25} - 1$
$= \frac{8}{25} - \frac{25}{25}$
$= -\frac{17}{25}$

b) $\tan 2\theta$

Substitute $\tan\theta = \frac{1}{2\sqrt{2}}$ in:

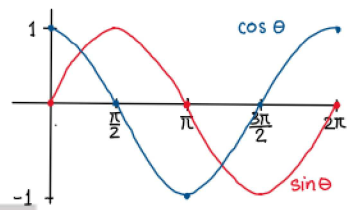
$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$= \frac{2\left(\frac{1}{2\sqrt{2}}\right)}{1 - \left(\frac{1}{2\sqrt{2}}\right)^2}$$

$$= \frac{\frac{1}{\sqrt{2}}}{\frac{7}{8}}, \text{ or } \frac{4\sqrt{2}}{7}$$

THINK FURTHER

In *Example 1b*, what other strategy could have been used to determine $\tan 2\theta$? Show that this strategy results in the same answer.



Example 2 Using Double-Angle Identities to Simplify and Evaluate

Write each expression as a single trigonometric ratio, then evaluate where possible.

a) $\sin\frac{\pi}{3}\cos\frac{\pi}{3}$ **b)** $6\cos^2\theta - 3$

SOLUTION

a) $\sin\frac{\pi}{3}\cos\frac{\pi}{3}$

Use: $2\sin\theta\cos\theta = \sin 2\theta$

Then, $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$

So, $\sin\frac{\pi}{3}\cos\frac{\pi}{3} = \frac{1}{2}\left(\sin 2\left(\frac{\pi}{3}\right)\right)$

$$= \frac{1}{2}\left(\sin\frac{2\pi}{3}\right)$$

$$= \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\sqrt{3}}{4}$$

b) $6\cos^2\theta - 3$

Use: $2\cos^2\theta - 1 = \cos 2\theta$

Then, $6\cos^2\theta - 3 = 3(2\cos^2\theta - 1)$

$$= 3\cos 2\theta$$

Check Your Understanding

2. Write each expression as a single trigonometric ratio, then evaluate where possible.

a) $\cos^2\left(\frac{\pi}{4}\right) - \sin^2\left(\frac{\pi}{4}\right)$

b) $\dfrac{2\tan\frac{\pi}{6}}{\tan^2\left(\frac{\pi}{6}\right) - 1}$ ← factor -1 to change the signs

a) $\cos\left(2\cdot\frac{\pi}{4}\right)$

$= \cos\left(\frac{\pi}{2}\right)$

$= 0$

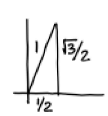
b) $\dfrac{2\tan\frac{\pi}{6}}{-1\left(1-\tan^2\left(\frac{\pi}{6}\right)\right)}$

$= -\tan\left(2\cdot\frac{\pi}{6}\right)$

$= -\tan\frac{\pi}{3}$

$= -\dfrac{\sqrt{3}/2}{1/2}$

$= -\sqrt{3}$



Example 3 Using Double-Angle Identities to Prove Other Identities

Prove each identity.

a) $\frac{1 - \cos 2\theta}{\sin 2\theta} = \tan\theta$ **b)** $\csc 2\theta + 1 = \frac{(\sin\theta + \cos\theta)^2}{\sin 2\theta}$

SOLUTION

a) $\frac{1 - \cos 2\theta}{\sin 2\theta} = \tan\theta$

$\text{L.S.} = \frac{1 - \cos 2\theta}{\sin 2\theta}$ Use identities for $\sin 2\theta$ and $\cos 2\theta$.

$= \frac{1 - (1 - 2\sin^2\theta)}{2\sin\theta\cos\theta}$

$= \frac{\cancel{2}\sin^{\cancel{2}}\theta}{\cancel{2}\cancel{\sin\theta}\cos\theta}$

$= \frac{\sin\theta}{\cos\theta}$

$= \tan\theta$

$= \text{R.S.}$

Since the left side is equal to the right side, the identity is proved.

b) $\csc 2\theta + 1 = \frac{(\sin\theta + \cos\theta)^2}{\sin 2\theta}$

$\text{R.S.} = \frac{(\sin\theta + \cos\theta)^2}{\sin 2\theta}$ Expand the numerator.

$= \frac{\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta}{\sin 2\theta}$ Use the Pythagorean identity and the identity for $\sin 2\theta$.

$= \frac{1 + \sin 2\theta}{\sin 2\theta}$

$= \frac{1}{\sin 2\theta} + \frac{\sin 2\theta}{\sin 2\theta}$

$= \csc 2\theta + 1$

$= \text{L.S.}$

Since the left side is equal to the right side, the identity is proved.

Check Your Understanding

3. Prove each identity.

a) $\cot\theta = \frac{\cos 2\theta + 1}{\sin 2\theta}$

b) $\cot\theta\csc 2\theta = \frac{1}{2\sin^2\theta}$

a) $RS = \frac{\cos 2\theta + 1}{\sin 2\theta}$

$= \frac{2\cos^2\theta - 1 + 1}{2\sin\theta\cos\theta}$

$= \frac{\cancel{2}\cos^{\cancel{2}}\theta}{\cancel{2}\sin\theta\cancel{\cos\theta}}$

$= \frac{\cos\theta}{\sin\theta}$

$= \cot\theta$

$= LS$

b) $LS = \cot\theta\csc 2\theta$

$= \frac{\cos\theta}{\sin\theta} \cdot \frac{1}{\sin 2\theta}$

$= \frac{\cancel{\cos\theta}}{\sin\theta} \cdot \frac{1}{2\sin\theta\cancel{\cos\theta}}$

$= \frac{1}{2\sin^2\theta}$

$= RS$

Try: p.658 #5,6,9-11

Example 4 Using Double-Angle Identities to Solve an Equation

Solve the equation $\cos 2x = 1 - 2\sin x$ over the domain $0 \leq x < 2\pi$.

SOLUTION

$\cos 2x = 1 - 2\sin x$ Use the identity for $\cos 2x$ that involves $\sin x$.

$1 - 2\sin^2 x = 1 - 2\sin x$

$2\sin^2 x - 2\sin x = 0$ Factor.

$2\sin x(\sin x - 1) = 0$

$2\sin x = 0$ or $\sin x - 1 = 0$

$\sin x = 0$ $\quad \sin x = 1$

$x = 0$ or $x = \pi$ $\quad x = \frac{\pi}{2}$

Verify the solution.

The roots are: $x = 0, x = \pi$, and $x = \frac{\pi}{2}$

Check Your Understanding

4. Solve the equation
$\frac{1}{2}\sin 2x - \cos^2 x = 0$
over the domain $0 \leq x < 2\pi$.

Discuss the Ideas

1. What is a double-angle identity? How is it different from double a trigonometric ratio?

2. Why are there three double-angle identities for $\cos 2\theta$ but only one identity for $\sin 2\theta$?

3. What are two identities for $\tan 2\theta$? Give an example of when it would be better to use each identity.