# 7.5 Sum and Difference Identities

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**FOCUS** **Prove identities using the sum and difference identities.**

### Get Started

In the following statements, $a$ and $b$ are real numbers. Determine some values of $a$ and $b$ for which each statement is true. Identify any identities. Justify your answers.

$3(a + b) = 3a + 3b$ $\qquad (a + b)^2 = a^2 + b^2$

$\log (a + b) = \log a + \log b$ $\qquad \sin (a + b) = \sin a + \sin b$

### Construct Understanding

Verify that each statement is true:

$\sin (30° + 60°) = \sin 30° \cos 60° + \cos 30° \sin 60°$

$\cos (30° + 60°) = \cos 30° \cos 60° - \sin 30° \sin 60°$

Determine similar statements for $\sin (30° - 60°)$ and $\cos (30° - 60°)$.



The statements that were verified above can be generalized to form the **sum and difference identities.**

**Sum and Difference Identities for Sine and Cosine**

$\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$

$\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$

$\sin (\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$

$\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$

The sum or difference identities can be used to determine the exact values of some sine and cosine ratios. For example, to determine the exact value of sin 75°, write 75° as 45° + 30°, then use:

$\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ Substitute: $\alpha = 45°, \beta = 30°$

$\sin (45° + 30°) = \sin 45° \cos 30° + \cos 45° \sin 30°$

$$\sin 75° = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3} + 1}{2\sqrt{2}}$$

**THINK FURTHER**

How else could you use a sum or difference identity to determine sin 75°?



**Example 1** **Applying the Sum and Difference Identities**

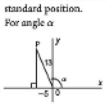
Given angle $\alpha$ in standard position with its terminal arm in Quadrant 2 and $\cos \alpha = -\frac{5}{13}$, and angle $\beta$ in standard position with its terminal arm in Quadrant 1 and $\sin \beta = \frac{3}{5}$, determine the exact value of $\cos (\alpha - \beta)$.

**SOLUTION**

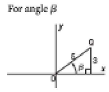
Use the identity: $\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$

To determine the values of $\cos \beta$ and $\sin \alpha$, sketch each angle in standard position.

For angle $\alpha$




For angle $\beta$

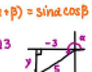



**Check Your Understanding**

1. Given angle $\alpha$ in standard position with its terminal arm in Quadrant 3 and $\cos \alpha = -\frac{3}{5}$, and angle $\beta$ in standard position with its terminal arm in Quadrant 2 and $\sin \beta = \frac{1}{3}$, determine the exact value of $\sin (\alpha + \beta)$.

$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$

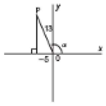
$\alpha$ : Q3




$(-3)^2 + y^2 = 5^2$

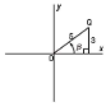
standard position.

For angle $\alpha$




Use: $x^2 + y^2 = r^2$
Substitute: $x = -5, r = 13$
$(-5)^2 + y^2 = 13^2$
$y^2 = 144$
$y = \pm 12$
Since the terminal arm of angle $\alpha$ lies in Quadrant 2, $y$ is positive.
So, $\sin \alpha = \frac{12}{13}$

For angle $\beta$




Use: $x^2 + y^2 = r^2$
Substitute: $y = 3, r = 5$
$x^2 + 3^2 = 5^2$
$x^2 = 16$
$x = \pm 4$
Since the terminal arm of angle $\beta$ lies in Quadrant 1, $x$ is positive.
So, $\cos \beta = \frac{4}{5}$

Substitute the values of the trigonometric ratios in the identity:

$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$

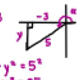
$\cos(\alpha - \beta) = \left(-\frac{5}{13}\right)\left(\frac{4}{5}\right) + \left(\frac{12}{13}\right)\left(\frac{3}{5}\right)$

$\cos(\alpha - \beta) = -\frac{20}{65} + \frac{36}{65}$

$\cos(\alpha - \beta) = \frac{16}{65}$

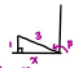
$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\alpha$ : Q3




$(-3)^2 + y^2 = 5^2$
$9 + y^2 = 25$
$y^2 = 16$
$y = -4$ ← in Q3
$\therefore \sin\alpha = \frac{-4}{5}$

$\beta$ : Q2




$x^2 + 1^2 = 3^2$
$x^2 + 1 = 9$
$x^2 = 8$
$x = -\sqrt{8} = -2\sqrt{2}$
$\therefore \cos\beta = \frac{-2\sqrt{2}}{3}$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
$= \left(-\frac{4}{5}\right)\left(-\frac{2\sqrt{2}}{3}\right) + \left(-\frac{3}{5}\right)\left(\frac{1}{3}\right)$
$= \frac{8\sqrt{2}}{15} + \frac{-3}{15}$
$= \frac{8\sqrt{2} - 3}{15}$



There are sum and difference identities for the tangent ratio.

$$\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} \quad \text{Expand.}$$

$$= \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta - \sin \alpha \sin \beta} \quad \text{Divide numerator and denominator by } \cos \alpha \cos \beta.$$

$$= \frac{\frac{\sin \alpha \cos \beta}{\cos \alpha \cos \beta} + \frac{\cos \alpha \sin \beta}{\cos \alpha \cos \beta}}{\frac{\cos \alpha \cos \beta}{\cos \alpha \cos \beta} - \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}} \quad \text{Simplify.}$$

$$= \frac{\frac{\sin \alpha}{\cos \alpha} + \frac{\sin \beta}{\cos \beta}}{1 - \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}} \quad \text{Use the tangent identity.}$$

$$= \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

**Sum and Difference Identities for Tangent**

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \qquad \tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

## Example 2 Using Sum and Difference Identities to Simplify and Evaluate

Write each expression in simplest form, then evaluate where possible.

**a)** $\sin 3\theta \sin \theta - \cos 3\theta \cos \theta$ **b)** $\dfrac{\tan \frac{\pi}{2} - \tan \frac{\pi}{3}}{1 + \tan \frac{\pi}{2} \tan \frac{\pi}{3}}$

**SOLUTION**

**a)** $\sin 3\theta \sin \theta - \cos 3\theta \cos \theta = -(\cos 3\theta \cos \theta - \sin 3\theta \sin \theta)$
Use the sum identity for cosine with $\alpha = 3\theta$ and $\beta = \theta$.
$\sin 3\theta \sin \theta - \cos 3\theta \cos \theta = -\cos(3\theta + \theta)$
$= -\cos 4\theta$
This simplest form cannot be evaluated because the angle contains a variable.

**Check Your Understanding**

**2.** Write each expression in simplest form, then evaluate where possible.

a) $\sin 8x \cos 3x - \cos 8x \sin 3x$

b) $\dfrac{\tan \frac{\pi}{6} + \tan \frac{\pi}{12}}{1 - \tan \frac{\pi}{6} \tan \frac{\pi}{12}}$

a) $\sin(8x - 3x)$
$= \sin(5x)$

b) $\tan\left(\frac{\pi}{6} + \frac{\pi}{12}\right)$
$= \tan\left(\frac{2\pi}{12} + \frac{\pi}{12}\right)$
$= \tan\left(\frac{3\pi}{12}\right)$
$= \tan\left(\frac{\pi}{4}\right)$
$= 1$

**b)** $\dfrac{\tan\frac{\pi}{2} - \tan\frac{\pi}{3}}{1 + \tan\frac{\pi}{2}\tan\frac{\pi}{3}}$

Use the difference identity for tangent with $\alpha = \frac{\pi}{2}$ and $\beta = \frac{\pi}{3}$.

$$\begin{aligned}\frac{\tan\frac{\pi}{2} - \tan\frac{\pi}{3}}{1 + \tan\frac{\pi}{2}\tan\frac{\pi}{3}} &= \tan\left(\frac{\pi}{2} - \frac{\pi}{3}\right)\\ &= \tan\frac{\pi}{6}\\ &= \frac{1}{\sqrt{3}}\end{aligned}$$

## Example 3 Using a Difference Identity to Prove an Identity

Prove this identity: $\cos\left(\frac{\pi}{2} - x\right) = \sin x$

**SOLUTION**

Use the difference identity, $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

Substitute: $\alpha = \frac{\pi}{2}, \beta = x$

$$\begin{aligned}\text{L.S.} &= \cos\left(\frac{\pi}{2} - x\right)\\ &= \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x\\ &= (0)\cos x + (1)\sin x\\ &= \sin x\\ &= \text{R.S.}\end{aligned}$$

Since the left side is equal to the right side, the identity is proved.

**Check Your Understanding**

**3.** Prove this identity:
$\sin(\pi - x) = \sin x$

$$\begin{aligned}LS &= \sin(\pi - x)\\ &= \sin\pi\cos x - \cos\pi\sin x\\ &= 0\cdot\cos x - (-1)\cdot\sin x\\ &= 0 + 1\sin x\\ &= \sin x\\ &= RS\end{aligned}$$

Try: #5-9,13,14

### THINK FURTHER

What geometric strategy could you use to verify the identity in *Example 3*, when $x$ is an acute angle?



## Example 4 Using the Sum and Difference Identities to Solve an Equation

Solve the equation $\sin 5x\cos 3x - \cos 5x\sin 3x = 1$ over the domain $0 \le x < 2\pi$.

**SOLUTION**

$$\begin{aligned}\sin 5x\cos 3x - \cos 5x\sin 3x &= 1\\ \sin(5x - 3x) &= 1\\ \sin 2x &= 1\end{aligned}$$

The given domain for angle $x$ is $0 \le x < 2\pi$,
so the domain for angle $2x$ is $0 \le 2x < 4\pi$.

$2x = \frac{\pi}{2}$ or $2x = \frac{5\pi}{2}$

$x = \frac{\pi}{4}$ $\quad x = \frac{5\pi}{4}$

The roots are: $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$

**Check Your Understanding**

**4.** Solve the equation
$\cos 4x\cos x + \sin 4x\sin x = 1$
over the domain $0 \le x < 2\pi$.

## Discuss the Ideas

**1.** How do you know whether you can use a sum or difference identity to determine the exact value of a trigonometric ratio of a given angle?

**2.** What strategies do you have for remembering the sum and difference formulas?