7.4 The Pythagorean Identities

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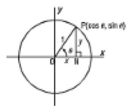
FOCUS Prove then apply the three Pythagorean identities.

Get Started

In the unit circle at the right, explain why $\sin^2\theta + \cos^2\theta = 1$. This equation is an identity. Why do you think it is called the Pythagorean identity?



$a^2 + b^2 = c^2$

$x^2 + y^2 = 1^2$

$\cos^2\theta + \sin^2\theta = 1$

$\sin\theta = \frac{y}{r} = \frac{y}{1} = y$

$\cos\theta = \frac{x}{r} = \frac{x}{1} = x$

$\sin x^2$

$(\sin x)^2 = \sin^2 x$

Construct Understanding

Prove algebraically that each equation below is an identity. Determine any non-permissible values of θ.

$\tan^2\theta + 1 = \sec^2\theta$ $\quad$ $1 + \cot^2\theta = \csc^2\theta$

$$\begin{aligned} LS &= \tan^2\theta + 1 \\ &= \frac{\sin^2\theta}{\cos^2\theta} + 1 \\ &= \frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} \\ &= \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta} \\ &= \frac{1}{\cos^2\theta} \\ &= \sec^2\theta \\ &= RS \end{aligned}$$

NPV: $\cos\theta \neq 0$

$\theta \neq \frac{(2n+1)\pi}{2}, n \in \mathbb{Z}$

$$\begin{aligned} LS &= 1 + \cot^2\theta \\ &= 1 + \frac{\cos^2\theta}{\sin^2\theta} \\ &= \frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} \\ &= \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta} \\ &= \frac{1}{\sin^2\theta} \\ &= \csc^2\theta \\ &= RS \end{aligned}$$

NPV: $\sin\theta \neq 0$

$\theta \neq \pi n, n \in \mathbb{Z}$

$\cos^2\theta = 1 - \sin^2\theta$

$\sin^2\theta = 1 - \cos^2\theta$

The following identities are the Pythagorean identities.

Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

These identities can be rearranged to form equivalent identities.

$\sin^2\theta + \cos^2\theta = 1$ can be written as:
$\sin^2\theta = 1 - \cos^2\theta$ or $\cos^2\theta = 1 - \sin^2\theta$

$\tan^2\theta + 1 = \sec^2\theta$ can be written as:
$\tan^2\theta = \sec^2\theta - 1$ or $\sec^2\theta - \tan^2\theta = 1$

$1 + \cot^2\theta = \csc^2\theta$ can be written as:
$\cot^2\theta = \csc^2\theta - 1$ or $\csc^2\theta - \cot^2\theta = 1$

The Pythagorean identities can be used to prove other identities or to simplify an equation before solving it.

Example 1 Using the Pythagorean Identities to Prove Other Identities

Prove each identity.

a) $\csc\theta\cos^2\theta + \sin\theta = \csc\theta$ $\quad$ b) $\sin^4\theta - \cos^4\theta = \sin^2\theta - \cos^2\theta$

SOLUTION

a) $\csc\theta\cos^2\theta + \sin\theta = \csc\theta$

$$\begin{aligned} \text{L.S.} &= \csc\theta\cos^2\theta + \sin\theta && \text{Replace } \csc\theta \text{ with } \frac{1}{\sin\theta}. \\ &= \frac{1}{\sin\theta}(\cos^2\theta) + \sin\theta \\ &= \frac{\cos^2\theta}{\sin\theta} + \sin\theta && \text{Use a common denominator.} \\ &= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta} && \text{Replace } \cos^2\theta + \sin^2\theta \text{ with 1.} \\ &= \frac{1}{\sin\theta} \end{aligned}$$

Check Your Understanding

1. Prove each identity.
   a) $\cot\theta + \tan\theta = \csc\theta\sec\theta$
   b) $\cot^3\theta = \cot\theta\csc^2\theta - \cot\theta$

a) $LS = \cot\theta + \tan\theta$

$= \frac{\cos\theta}{\sin\theta} \cdot \frac{\cos\theta}{\cos\theta} + \frac{\sin\theta}{\cos\theta} \cdot \frac{\sin\theta}{\sin\theta}$

$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}$

$= \frac{1}{\sin\theta\cos\theta}$

$= \frac{1}{\sin\theta}(\cos^2\theta) + \sin\theta$

$= \frac{\cos^2\theta}{\sin\theta} + \sin\theta$ Use a common denominator.

$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta}$ Replace $\cos^2\theta + \sin^2\theta$ with 1.

$= \frac{1}{\sin\theta}$

$= \csc\theta$

$=$ R.S.

The left side is equal to the right side, so the identity is proved.

$= \frac{\cos\theta}{\sin\theta}\cdot\frac{\cos\theta}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\cdot\frac{\sin\theta}{\sin\theta}$

$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}$

$= \frac{1}{\sin\theta\cos\theta}$

$= \csc\theta\sec\theta$

$=$ RS

b) $\sin^4\theta - \cos^4\theta = \sin^2\theta - \cos^2\theta$

L.S. $= \sin^4\theta - \cos^4\theta$ Factor using a difference of squares.

$= (\sin^2\theta - \cos^2\theta)(\sin^2\theta + \cos^2\theta)$ Replace $\sin^2\theta + \cos^2\theta$ with 1.

$= \sin^2\theta - \cos^2\theta$

$=$ R.S.

The left side is equal to the right side, so the identity is proved.

b) RS $= \cot\theta\csc^2\theta - \cot\theta$

$= \cot\theta(\csc^2\theta - 1)$

$= \cot\theta(\cot^2\theta)$

$= \cot^3\theta$

$=$ LS

Try 6ii, 7ii

THINK FURTHER

How could you prove the identity in *Example 1b*, by beginning with the right side?

Example 1 illustrates these strategies to help to prove an identity:

- writing terms with a common denominator
- factoring using the difference of squares

Recall how to rationalize a binomial denominator, such as $\frac{2}{1 + \sqrt{3}}$, by multiplying the numerator and denominator by the conjugate $1 - \sqrt{3}$:

$$\left(\frac{2}{1 + \sqrt{3}}\right)\left(\frac{1 - \sqrt{3}}{1 - \sqrt{3}}\right) = \frac{2 - 2\sqrt{3}}{-2}$$

$$= \frac{-2(\sqrt{3} - 1)}{-2}$$

$$= \sqrt{3} - 1$$

This strategy may also be used in the proofs for some trigonometric identities that involve factors such as $1 \pm \sin\theta$ or $1 \pm \cos\theta$.

Example 2 Proving Identities Involving Fractions

Prove each identity.

a) $\frac{\cos\theta\sin\theta}{1 + \sin\theta} = \frac{1 - \sin\theta}{\cot\theta}$ **b)** $2\sec\theta = \frac{\cos\theta}{1 - \sin\theta} + \frac{\cos\theta}{1 + \sin\theta}$

SOLUTION

a) $\frac{\cos\theta\sin\theta}{1 + \sin\theta} = \frac{1 - \sin\theta}{\cot\theta}$

L.S. $= \frac{\cos\theta\sin\theta}{1 + \sin\theta}$ Multiply numerator and denominator by the conjugate of the denominator.

$= \left(\frac{\cos\theta\sin\theta}{1 + \sin\theta}\right)\left(\frac{1 - \sin\theta}{1 - \sin\theta}\right)$

$= \frac{\cos\theta\sin\theta(1 - \sin\theta)}{1 - \sin^2\theta}$

$= \frac{\cancel{\cos\theta}\sin\theta(1 - \sin\theta)}{\cos^{\cancel{2}}\theta}$ Replace $\frac{\sin\theta}{\cos\theta}$ with $\tan\theta$.

$= \tan\theta(1 - \sin\theta)$ Replace $\tan\theta$ with $\frac{1}{\cot\theta}$.

$= \frac{1 - \sin\theta}{\cot\theta}$

$=$ R.S.

The left side is equal to the right side, so the identity is proved.

b) $2\sec\theta = \frac{\cos\theta}{1 - \sin\theta} + \frac{\cos\theta}{1 + \sin\theta}$

R.S. $= \frac{\cos\theta}{1 - \sin\theta} + \frac{\cos\theta}{1 + \sin\theta}$ Write the fractions with a common denominator.

$= \frac{\cos\theta(1 + \sin\theta) + \cos\theta(1 - \sin\theta)}{(1 - \sin\theta)(1 + \sin\theta)}$ Expand.

$= \frac{\cos\theta + \cos\theta\sin\theta + \cos\theta - \cos\theta\sin\theta}{1 - \sin^2\theta}$ Replace $1 - \sin^2\theta$ with $\cos^2\theta$.

$= \frac{2\cancel{\cos\theta}}{\cos^{\cancel{2}}\theta}$

$= \frac{2}{\cos\theta}$

$= 2\sec\theta$

$=$ L.S.

The left side is equal to the right side, so the identity is proved.

Check Your Understanding

2. Prove each identity.

a) $\frac{1 - \cos\theta}{\sin\theta} = \frac{\sin\theta}{1 + \cos\theta}$

b) $\frac{1}{1 - \cos\theta} + \frac{1}{1 + \cos\theta} = 2\csc^2\theta$

a) Start with the side which has two terms in the denominator

RS $= \frac{\sin\theta}{1 + \cos\theta}\cdot\frac{1 - \cos\theta}{1 - \cos\theta}$

$= \frac{\sin\theta(1 - \cos\theta)}{(1 + \cos\theta)(1 - \cos\theta)}$

$= \frac{\sin\theta(1 - \cos\theta)}{1 - \cos^2\theta}$

$= \frac{\cancel{\sin}\theta(1 - \cos\theta)}{\sin^{\cancel{2}}\theta}$

$= \frac{1 - \cos\theta}{\sin\theta}$

$=$ LS

b) LS $= \frac{1\,(1+\cos\theta)}{(1 - \cos\theta)\cdot(1+\cos\theta)} + \frac{1\,(1-\cos\theta)}{(1 + \cos\theta)\cdot(1-\cos\theta)}$

$= \frac{1 + \cos\theta + 1 - \cos\theta}{(1 - \cos\theta)(1 + \cos\theta)}$

$= \frac{2}{1 - \cos^2\theta}$

$= \frac{2}{\sin^2\theta}$

$= 2\csc^2\theta$

$=$ RS

Example 3 Using the Pythagorean Identities in the Solution of an Equation

Use algebra to solve the equation $2\cos^2 x - 3\sin x = 0$ over the domain $\frac{\pi}{2} \leq x < 2\pi$.

SOLUTION

$2\cos^2 x - 3\sin x = 0$ Replace $\cos^2 x$ with $1 - \sin^2 x$.

$2(1 - \sin^2 x) - 3\sin x = 0$

$2 - 2\sin^2 x - 3\sin x = 0$

$2\sin^2 x + 3\sin x - 2 = 0$ Factor.

$(2\sin x - 1)(\sin x + 2) = 0$

Either $2\sin x - 1 = 0$

$\sin x = \frac{1}{2}$

$x = \frac{5\pi}{6}$

or $\sin x + 2 = 0$

$\sin x = -2$

Since -2 is outside the range of $\sin x$, there is no solution.

So, the root is: $x = \frac{5\pi}{6}$

Verify the root by substitution.

Check Your Understanding

3. Use algebra to solve the equation
$3 - 3\cos x - 2\sin^2 x = 0$
over the domain $0 \leq x < \frac{3\pi}{2}$.

$3 - 3\cos x - 2(1 - \cos^2 x) = 0$

$3 - 3\cos x - 2 + 2\cos^2 x = 0$

$2\cos^2 x - 3\cos x + 1 = 0$

$(2\cos x - 1)(\cos x - 1) = 0$

$\cos x = \frac{1}{2}$ $\cos x = 1$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$ $x = 0$

not in domain

$\therefore x = 0, \frac{\pi}{3}$

Try: 6ii, 7ii, 10 (in radians), 12a

Discuss the Ideas

1. When can you use a conjugate to help prove an identity?

2. The Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ was shown to be true for an angle θ in Quadrant 1 in Grade 11. How do you know that this identity and the other two Pythagorean identities are true for all values of θ for which the trigonometric ratios are defined?