7.3 Reciprocal and Quotient Identities

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FOCUS Use the reciprocal and quotient identities to prove other identities and to solve trigonometric equations.

Get Started

Simplify each expression, where $a, b, c, d \neq 0$

$\frac{a}{b} + \frac{b}{c} = \frac{ac + b^2}{bc}$

$\frac{a^2 - ab}{b^2 - a^2} = \frac{a(a-b)}{(b-a)(b+a)}$

$= \frac{-a\cancel{(b-a)}}{\cancel{(b-a)}(b+a)}$

$= \frac{-a}{b+a}$

$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \div \frac{c}{d}$

$= \frac{a}{b} \times \frac{d}{c}$

$= \frac{ad}{bc}$

$\frac{\frac{a}{c} + \frac{1}{b}}{b + \frac{c}{a}} = \left(\frac{a}{c} + \frac{1}{b}\right) \div \left(b + \frac{c}{a}\right)$

$= \left(\frac{ab + c}{bc}\right) \div \left(\frac{ab + c}{a}\right)$

$= \left(\frac{\cancel{ab + c}}{bc}\right) \times \left(\frac{a}{\cancel{ab + c}}\right)$

$= \frac{a}{bc}$

Construct Understanding

Use these values: $\theta = 0, \theta = \frac{\pi}{4}, \theta = \frac{\pi}{3}, \theta = \frac{\pi}{2}$

For which values of θ is each equation below true?

Is either equation true for all values of θ for which the trigonometric ratios are defined? Justify your response.

$\tan\theta\cos\theta = \sin\theta$ $\qquad$ $\tan\theta\sin\theta = \cos\theta$

What is the ratio of $\frac{\sin\theta}{\cos\theta}$?

$\sin\theta = \frac{y}{r}$ $\quad$ $\cos\theta = \frac{x}{r}$

$\frac{\sin\theta}{\cos\theta} = \frac{y/r}{x/r} = \frac{y}{x} = \tan\theta$

A **trigonometric identity** is a statement that relates trigonometric ratios, and is true for all values of the variable for which the trigonometric ratios are defined.

The equation $\tan\theta\cos\theta = \sin\theta$ is an identity because it is true for all values of θ except $\theta = \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$, where $\tan\theta$ is not defined.

In Lesson 6.1, six trigonometric functions were defined in terms of an angle θ in standard position in a unit circle and a terminal point $P(x, y)$ on the unit circle.

$\sin\theta = y$ $\qquad$ $\cos\theta = x$ $\qquad$ $\tan\theta = \frac{y}{x}, x \neq 0$

$\csc\theta = \frac{1}{y}, y \neq 0$ $\qquad$ $\sec\theta = \frac{1}{x}, x \neq 0$ $\qquad$ $\cot\theta = \frac{x}{y}, y \neq 0$

The following identities are developed from the definitions above.

Reciprocal Identities

$\csc\theta = \frac{1}{\sin\theta}$, $\sin\theta \neq 0$ $\qquad$ $\sec\theta = \frac{1}{\cos\theta}$, $\cos\theta \neq 0$ $\qquad$ $\cot\theta = \frac{1}{\tan\theta}$, $\sin\theta \neq 0$ and $\cos\theta \neq 0$

Quotient Identities

$\tan\theta = \frac{\sin\theta}{\cos\theta}, \cos\theta \neq 0$ $\qquad$ $\cot\theta = \frac{\cos\theta}{\sin\theta}, \sin\theta \neq 0$

The reciprocal identities can be rearranged and written in other forms:

$\csc\theta = \frac{1}{\sin\theta}, \sin\theta \neq 0$ $\qquad$ $\sec\theta = \frac{1}{\cos\theta}, \cos\theta \neq 0$

or, $\csc\theta\sin\theta = 1$ $\qquad$ or, $\sec\theta\cos\theta = 1$

$\cot\theta = \frac{1}{\tan\theta}, \sin\theta \neq 0$ and $\cos\theta \neq 0$

or, $\cot\theta\tan\theta = 1$

The restrictions on page 606 indicate that the expressions on both sides of an identity are equal for all values of the variable for which each expression is defined. This is true for all identities, so it is not necessary to write the non-permissible values of an identity because they can be identified by determining where each side of the identity is undefined.

A trigonometric identity can be verified algebraically by substituting a value for the variable. A verification shows that an identity is true for specific values of the variable; it does not prove that the identity is true for all values of the variable for which each trigonometric ratio is defined.

To prove an identity is valid, it must be shown that one side of the identity is equal to the other side, or that both sides are equal to the same expression.

When proving an identity, it is often helpful to start by writing all the trigonometric ratios in the identity in terms of the sine and cosine ratios to identify what can be simplified.

Example 1 **Verifying then Proving an Identity**

For each identity below:

i) Verify the identity for $\theta = \frac{\pi}{4}$.

ii) Prove the identity.

a) $\sin\theta\sec\theta\cot\theta = 1$ **b)** $\frac{\sin\theta + \cos\theta}{\sin\theta} = 1 + \cot\theta$

SOLUTION

a) i) $\sin\theta\sec\theta\cot\theta = 1$ Substitute: $\theta = \frac{\pi}{4}$

$$\text{L.S.} = \sin\frac{\pi}{4}\sec\frac{\pi}{4}\cot\frac{\pi}{4} \qquad \text{R.S.} = 1$$
$$= \left(\frac{1}{\sqrt{2}}\right)(\sqrt{2})(1)$$
$$= 1$$

The left side is equal to the right side, so $\theta = \frac{\pi}{4}$ is verified.

ii) Simplify the left side.

$$\text{L.S.} = \sin\theta\sec\theta\cot\theta \qquad \text{Substitute: } \sec\theta = \frac{1}{\cos\theta}, \cot\theta = \frac{\cos\theta}{\sin\theta}$$
$$= \cancel{\sin\theta}^{1}\cdot\frac{1}{\cancel{\cos\theta}_{1}}\cdot\frac{\cancel{\cos\theta}^{1}}{\cancel{\sin\theta}_{1}}$$
$$= 1$$
$$= \text{R.S.}$$

The left side is equal to the right side, so the identity is proved.

Check Your Understanding

1. For each identity below:

i) Verify the identity for $\theta = 30°$.

ii) Prove the identity.

a) $(\sec\theta)(1 + \cos\theta) = 1 + \sec\theta$

b) $1 - \tan\theta = \frac{\cot\theta - 1}{\cot\theta}$

a) $LS = (\sec\theta)(1+\cos\theta)$
$= \sec\theta + \sec\theta\cos\theta$
$= \sec\theta + \frac{1}{\cos\theta}\cdot\cos\theta$
$= \sec\theta + 1$
$= 1 + \sec\theta$
$= RS$

b) $RS = \frac{\cot\theta - 1}{\cot\theta}$
$= \frac{\cot\theta}{\cot\theta} - \frac{1}{\cot\theta}$
$= 1 - \tan\theta$
$= LS$

b) i) $\frac{\sin\theta + \cos\theta}{\sin\theta} = 1 + \cot\theta$ Substitute: $\theta = \frac{\pi}{4}$

$$\text{L.S.} = \frac{\sin\frac{\pi}{4} + \cos\frac{\pi}{4}}{\sin\frac{\pi}{4}} \qquad \text{R.S.} = 1 + \cot\frac{\pi}{4} = 1 + 1 = 2$$
$$= \frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$$
$$= \frac{\frac{2}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$$
$$= 2$$

The left side is equal to the right side, so $\theta = \frac{\pi}{4}$ is verified.

ii) Simplify the right side.

$$\text{R.S.} = 1 + \cot\theta \qquad \text{Substitute: } \cot\theta = \frac{\cos\theta}{\sin\theta}$$
$$= 1 + \frac{\cos\theta}{\sin\theta}$$
$$= \frac{\sin\theta + \cos\theta}{\sin\theta}$$
$$= \text{L.S.}$$

The left side is equal to the right side, so the identity is proved.

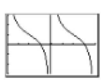
THINK FURTHER

In *Example 1a*, why did it make sense to start with the left side?

In *Example 1b*, the identity can also be proved by beginning with the left side of the identity:

$$\text{L.S.} = \frac{\sin\theta + \cos\theta}{\sin\theta}$$
$$= \frac{\sin\theta}{\sin\theta} + \frac{\cos\theta}{\sin\theta}$$
$$= 1 + \cot\theta$$
$$= \text{R.S.}$$

An identity can be verified graphically by graphing the functions that correspond to each side of the identity. For *Example 1b*, graph $y = \frac{\sin\theta + \cos\theta}{\sin\theta}$ and $y = 1 + \cot\theta$; notice that $\cot\theta$ is entered as $\frac{1}{\tan\theta}$.



The graphs coincide, so the identity is verified.

When both sides of an identity are complicated, it may be easier to prove the identity by showing that both sides simplify to the same expression. This is illustrated in *Example 2b*.

Example 2 **Identifying Restrictions then Proving an Identity**

For each identity below:

i) Determine the non-permissible values of θ.

ii) Prove the identity.

a) $\cos^2\theta = \frac{\cot\theta\sin\theta}{\sec\theta}$ **b)** $\frac{\sin\theta + \tan\theta}{1 + \cos\theta} = \sin\theta\sec\theta$

SOLUTION

a) i) $\cos^2\theta = \frac{\cot\theta\sin\theta}{\sec\theta}$

$\sec\theta = \frac{1}{\cos\theta}$, $\cot\theta = \frac{\cos\theta}{\sin\theta}$, and $\sec\theta$ is a denominator that is never 0, so the non-permissible values occur when:

$\cos\theta = 0$ or $\sin\theta = 0$

$\theta = \frac{\pi}{2} + \pi k, k \in Z$ $\quad\theta = \pi k, k \in Z$

These non-permissible values can be combined and written as: $\theta = \frac{\pi}{2}k, k \in Z$

Check Your Understanding

2. For each identity below:

i) Determine the non-permissible values of θ.

ii) Prove the identity.

a) $\frac{\cot\theta}{\csc\theta} = \cos\theta$

b) $\cos\theta = \frac{1 + \cos\theta}{1 + \sec\theta}$

a) $\csc\theta \neq 0$

csc θ and sec θ are never zero

$\cot\theta = \frac{\cos\theta}{\sin\theta} \rightarrow \sin\theta \neq 0$

$\theta = 0, \pi, 2\pi\ldots$

$\therefore \theta \neq \pi n, n \in Z$

$\sec\theta = \frac{1}{\cos\theta}$, $\cot\theta = \frac{\cos\theta}{\sin\theta}$, and $\sec\theta$ is a denominator that is never 0, so the non-permissible values occur when:

$\cos\theta = 0$ or $\sin\theta = 0$

$\theta = \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$ $\quad$ $\theta = \pi k, k \in \mathbb{Z}$

These non-permissible values can be combined and written as:
$\theta = \frac{\pi}{2}k, k \in \mathbb{Z}$

ii) Simplify the right side.

$$\text{R.S.} = \frac{\cot\theta\sin\theta}{\sec\theta} \qquad \text{Substitute: } \cot\theta = \frac{\cos\theta}{\sin\theta}, \sec\theta = \frac{1}{\cos\theta}$$

$$= \frac{\frac{\cos\theta}{\cancel{\sin\theta}} \cdot \cancel{\sin\theta}}{\frac{1}{\cos\theta}}$$

$$= \cos\theta\left(\frac{\cos\theta}{1}\right)$$

$$= \cos^2\theta$$

$$= \text{L.S.}$$

The left side is equal to the right side, so the identity is proved.

b) i) $\frac{\sin\theta + \tan\theta}{1 + \cos\theta} = \sin\theta\sec\theta$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\sec\theta = \frac{1}{\cos\theta}$, and $1 + \cos\theta$ is a denominator, so the non-permissible values occur when:

$\cos\theta = 0$ or $1 + \cos\theta = 0$

$\theta = \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$ $\quad$ $\cos\theta = -1$

$\theta = \pi + 2\pi k, k \in \mathbb{Z}$

cscθ and secθ are never zero

$\cot\theta = \frac{\cos\theta}{\sin\theta} \to \sin\theta \neq 0$

$\theta = 0, \pi, 2\pi\ldots$

$\therefore \theta \neq \pi n, n \in \mathbb{Z}$

$LS = \frac{\cot\theta}{\csc\theta}$

$= \frac{\cos\theta/\sin\theta}{1/\sin\theta}$

$= \cos\theta$

$= RS$

b) $\sec\theta \neq -1$

$\cos\theta \neq -1 \to \theta \neq \pi, 3\pi, 5\pi\ldots$

$\therefore \theta \neq (2n+1)\pi, n \in \mathbb{Z}$

$RS = \frac{1+\cos\theta}{1+\sec\theta}$

$= \frac{1+\cos\theta}{1+\frac{1}{\cos\theta}}$

$= \frac{\cos\theta(\frac{1}{\cos\theta}+1)}{1+\frac{1}{\cos\theta}}$

$= \cos\theta$

$= LS$

Try #3, 6

ii) Simplify the left side.

$$\text{L.S.} = \frac{\sin\theta + \tan\theta}{1 + \cos\theta}$$

Substitute: $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$$\text{L.S.} = \frac{\sin\theta + \frac{\sin\theta}{\cos\theta}}{1 + \cos\theta}$$

$$= \frac{\cos\theta\sin\theta + \sin\theta}{\cos\theta(1 + \cos\theta)}$$

$$= \frac{\sin\theta\cancel{(\cos\theta + 1)}}{\cos\theta\cancel{(1 + \cos\theta)}}$$

$$= \frac{\sin\theta}{\cos\theta}$$

Simplify the right side.

$$\text{R.S.} = \sin\theta\sec\theta$$

$$= \sin\theta\left(\frac{1}{\cos\theta}\right)$$

$$= \frac{\sin\theta}{\cos\theta}$$

The left and right sides simplify to the same expression, so the identity is proved.

Example 3 Using Identities in the Solution of an Equation

Use algebra to solve each equation over the domain $0 \leq x < 2\pi$. Give the roots to the nearest hundredth where necessary.

a) $3\cos x + 1 = 2\sec x$ $\qquad$ **b)** $\sin x + \sqrt{3}\cos x = 0$

SOLUTION

a) $3\cos x + 1 = 2\sec x$ $\qquad$ Substitute: $\sec x = \frac{1}{\cos x}$

$3\cos x + 1 = 2\left(\frac{1}{\cos x}\right)$ $\qquad$ Multiply each side by $\cos x$.

$3\cos^2 x + \cos x = 2$ $\qquad$ Solve for x.

$3\cos^2 x + \cos x - 2 = 0$

$(3\cos x - 2)(\cos x + 1) = 0$

$3\cos x - 2 = 0$ or $\cos x + 1 = 0$

$\cos x = \frac{2}{3}$ $\qquad$ $\cos x = -1$

$x \doteq 0.84$ or $x \doteq 5.44$ $\qquad$ $x = \pi$

The roots are: $x \doteq 0.84$, $x \doteq 5.44$, and $x = \pi$

Verify the roots by substitution.

b) $\sin x + \sqrt{3}\cos x = 0$ $\qquad$ Assume $\cos x \neq 0$, then divide by $\cos x$.

$\frac{\sin x}{\cos x} + \frac{\sqrt{3}\cos x}{\cos x} = 0$

$\tan x + \sqrt{3} = 0$

$\tan x = -\sqrt{3}$

$x = \frac{2\pi}{3}$ or $x = \frac{5\pi}{3}$

Check Your Understanding

3. Use algebra to solve each equation over the domain $0 \leq x < 2\pi$.

a) $2\sin x = 3 + 2\csc x$

b) $\sin x = \cos x$

a) $2\sin x = 3 + 2 \cdot \frac{1}{\sin x}$

multiply both sides by $\sin x$

$2\sin^2 x = 3\sin x + 2$

$2\sin^2 x - 3\sin x - 2 = 0$

$(2\sin x + 1)(\sin x - 2) = 0$

$\sin x = -\frac{1}{2}$ $\quad$ $\sin x = 2$ (never)

$\boxed{x = \frac{7\pi}{6}, \frac{11\pi}{6}}$

b) $\frac{\sin x}{\cos x} = \frac{\cos x}{\cos x}$

$\tan x = 1$

$\boxed{x = \frac{\pi}{4}, \frac{5\pi}{4}}$

Try #7-9(ii), 10, 11

For $\cos x = 0, x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$

Verify by substitution that neither value of x is a root of the given equation.

The roots are: $x = \frac{2\pi}{3}$ and $x = \frac{5\pi}{3}$

Verify the roots by substitution.

Here are some strategies to use to prove an identity:

- Start by simplifying the side of the identity that is more complex.
- Write the expressions in terms of $\sin x$ and $\cos x$.

Discuss the Ideas

1. What is the difference between a trigonometric identity and a trigonometric equation? Suppose you are given a trigonometric equation. What strategy can you use to check whether it might be an identity?

2. Can you conclude that an equation is an identity when it is shown to be valid for a given value of the variable? Explain.

Exercises

A

3. Write each expression in terms of a single trigonometric function.

a) $\frac{\cos\theta}{\sin\theta}$ b) $\frac{\sin^2\theta}{\cos^2\theta}$