7.2 Solving Trig Equations

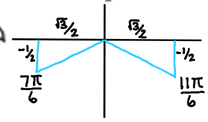
7.2 Solving Trigonometric Equations Algebraically

FOCUS Use an algebraic method to solve first- and second-degree trigonometric equations.

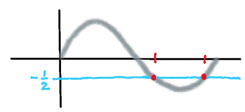
Get Started

$0 \le \theta < 2\pi$

Solve the equation: $\sin\theta = -\frac{1}{2}$ for $\cancel{0°} \le \theta < 360°$



$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

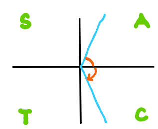


Construct Understanding

Without graphing, solve the equation $4\cos x + 3 = 7\cos x + 2$ for $0 \le x < 2\pi$.

Check the solution graphically.

$$4\cos x + 3 = 7\cos x + 2$$
$$-7\cos x \quad -3 \qquad -7\cos x \quad -3$$
$$-3\cos x = -1$$
$$\cos x = \frac{1}{3}$$
$$x = \cos^{-1}\left(\frac{1}{3}\right)$$
$$x \doteq 1.23 \leftarrow Q1$$



*There is another solution in Q4.

→ for cosine solutions, subtract answer from 2π

$2\pi - 1.23 \doteq 5.05$

$$\boxed{x = 1.23, 5.05}$$

To solve a first-degree trigonometric equation, isolate the trigonometric function to reduce the equation to the form $\sin x = a$, $\cos x = a$, or $\tan x = a$, where a is a constant. Some of these equations have exact numerical solutions. In these cases, the solution is a multiple of $\frac{\pi}{6}$ or $\frac{\pi}{4}$.

Consider solving this equation for $0 \le x < 2\pi$:

$2 \cos x + 1 = 0$ Isolate $\cos x$.

$$2 \cos x = -1$$

$$\cos x = -\frac{1}{2}$$

Use the completed table in Chapter 6, page 476, to identify the exact angle.

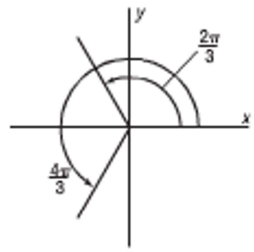
The cosine of an angle is negative when its terminal arm lies in Quadrant 2 or 3.

The reference angle is: $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

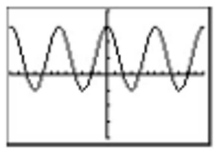
In Quadrant 2, x is: $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$

In Quadrant 3, x is: $\pi + \frac{\pi}{3} = \frac{4\pi}{3}$

So, the roots are: $x = \frac{2\pi}{3}$ and $x = \frac{4\pi}{3}$



The graph of $y = 2 \cos x + 1$ has period 2π.



So, the general solution of the equation $2 \cos x + 1 = 0$ is:

$x = \frac{2\pi}{3} + 2\pi k$ or $x = \frac{4\pi}{3} + 2\pi k$, where $k \in \mathbb{Z}$

THINK FURTHER

What are the roots of $2 \cos x + 1 = 0$ over the domain $-2\pi \le x < 0$?

Example 1 — Determining the Exact Roots of a First-Degree Trigonometric Equation

a) Use algebra to solve the equation $\sqrt{2}\sin x - 3 = -2$ for $-360° < x \leq 0°$, then write the general solution.

b) Use algebra to determine the general solution of the equation $\cos 3x = -\frac{1}{2}$ over the set of real numbers, then list the roots in the domain $-\pi \leq x < \pi$.

SOLUTION

a) $\sqrt{2}\sin x - 3 = -2$ Solve for $\sin x$.

$\sqrt{2}\sin x = 1$

$\sin x = \frac{1}{\sqrt{2}}$

Since $\sin x$ is positive, the terminal arm of angle x lies in Quadrant 1 or 2.

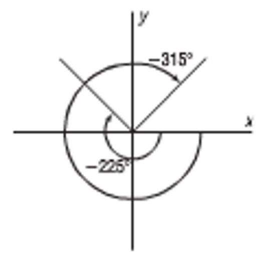
The reference angle is: $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$

In Quadrant 1, $x = -360° + 45°$

$x = -315°$

In Quadrant 2, $x = -180° - 45°$

$x = -225°$



The roots are: $x = -315°$ and $x = -225°$

The period of $\sin x$ is 360°, so the general solution is:

$x = -315° + k360°, k \in \mathbb{Z}$

or $x = -225° + k360°, k \in \mathbb{Z}$

b) $\cos 3x = -\frac{1}{2}$

Since $\cos 3x$ is negative, the terminal arm of angle $3x$ lies in Quadrant 2 or 3.

The reference angle for angle $3x$ is: $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

In Quadrant 2:	In Quadrant 3:
A solution is:	A solution is:
$3x = \pi - \frac{\pi}{3}$	$3x = \pi + \frac{\pi}{3}$
$3x = \frac{2\pi}{3}$	$3x = \frac{4\pi}{3}$
$x = \frac{2\pi}{9}$	$x = \frac{4\pi}{9}$

Check Your Understanding

1. a) Use algebra to solve the equation $7 + 2\sin x = 4\sin x + 5$ for -2π $\cancel{-360°}$ $< x \leq 0$, then write the general solution.

b) Use algebra to determine the general solution of the equation $\cos 3x = -1$ over the set of real numbers, then list the roots in the domain $-2\pi \leq x < 0$.

a) $7 + 2\sin x = 4\sin x + 5$

$-2\sin x = -2$

$\sin x = 1$

$x = \frac{\pi}{2}$

* but $-2\pi < x \leq 0$ so subtract 2π

$\frac{\pi}{2} - 2\pi = -\frac{3\pi}{2}$

general solution: $x = -\frac{3\pi}{2} + 2\pi n$ where $n \in \mathbb{Z}$

b) $\cos 3x = -1$ $\cos x = -1$

$3x = \pi$ $x = \pi$

$x = \frac{\pi}{3}$

$\cos 3x$ has a period of $\frac{2\pi}{3}$

$\therefore$ other solutions in $0 \leq x < 2\pi$ are

$\frac{\pi}{3} + \frac{2\pi}{3} = \frac{3\pi}{3} = \pi$

$\pi + \frac{2\pi}{3} = \frac{5\pi}{3}$

general solution: $x = \frac{\pi}{3} + \frac{2\pi}{3}n$ where $n \in \mathbb{Z}$

Try #2,4,7

#2 $\sin x = \pm\frac{1}{2}$ $\tan x = \pm 1, 0$

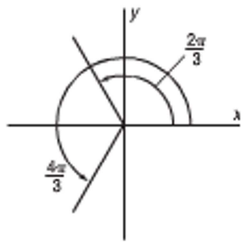
$\sin x = \pm\frac{\sqrt{3}}{2}$ $\tan x = \pm\frac{1}{\sqrt{3}}$

$\sin x = \pm\frac{1}{\sqrt{2}}$ $\tan x = \pm\sqrt{3}$

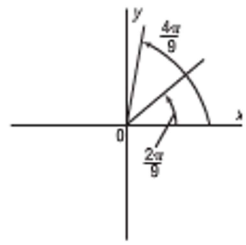
$\sin x = 0, \pm 1$

(same for cosine)

For angle $3x$



For angle x



The period of $\cos 3x$ is $\frac{2\pi}{3}$, so the general solution is:

$x = \frac{2\pi}{9} + \frac{2\pi}{3}k, k \in \mathbb{Z}$ or $x = \frac{4\pi}{9} + \frac{2\pi}{3}k, k \in \mathbb{Z}$

Substitute integer values for k to obtain all other roots between $-\pi$ and π.

When $k = 1$:

$x = \frac{2\pi}{9} + \frac{2\pi}{3}$ or $x = \frac{4\pi}{9} + \frac{2\pi}{3}$

$x = \frac{8\pi}{9}$ $\quad$ $x = \frac{10\pi}{9}$, not in the domain

When $k = -1$:

$x = \frac{2\pi}{9} - \frac{2\pi}{3}$ or $x = \frac{4\pi}{9} - \frac{2\pi}{3}$

$x = -\frac{4\pi}{9}$ $\quad$ $x = -\frac{2\pi}{9}$

When $k = -2$:

$x = \frac{2\pi}{9} - \frac{4\pi}{3}$ or $x = \frac{4\pi}{9} - \frac{4\pi}{3}$

$x = -\frac{10\pi}{9}$, not in the domain $\quad$ $x = -\frac{8\pi}{9}$

The roots are: $x = \pm\frac{2\pi}{9}$, $x = \pm\frac{4\pi}{9}$, and $x = \pm\frac{8\pi}{9}$

In *Example 1*, each root can be verified by substituting it into the original equation.

Example 2 Determining the Approximate Roots of a First-Degree Trigonometric Equation

a) To the nearest hundredth, solve the equation $5 - 3\tan x = 2\tan x + 1$ for $-\pi \leq x < \frac{3\pi}{2}$, then write the general solution.

b) To the nearest degree, solve the equation $3 = \sqrt{3}\sec x$ for $-90° \leq x < 270°$, then write the general solution.

SOLUTION

a) $5 - 3\tan x = 2\tan x + 1$ Solve for $\tan x$.

$4 = 5\tan x$

$\tan x = \frac{4}{5}$

Since $\tan x$ is positive, the terminal arm of angle x lies in Quadrant 1 or 3.

The reference angle is: $\tan^{-1}\left(\frac{4}{5}\right) = 0.6747\ldots$

For $0 \leq x < \frac{3\pi}{2}$:

In Quadrant 1, $x = 0.6747\ldots$

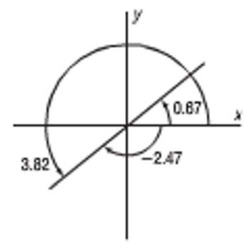
In Quadrant 3, $x = \pi + 0.6747\ldots$

$x = 3.8163\ldots$

For $-\pi \leq x < 0$:

In Quadrant 3, $x = -\pi + 0.6747\ldots$

$x = -2.4668\ldots$



To the nearest hundredth, the roots are:
$x = 0.67$, $x = 3.82$, and $x = -2.47$

The period of $\tan x$ is π, so the general solution is approximately:
$x = 0.67 + \pi k$, $k \in \mathbb{Z}$

b) $3 = \sqrt{3}\sec x$ Solve for $\cos x$.

$\sec x = \frac{3}{\sqrt{3}}$, or $\sqrt{3}$

$\cos x = \frac{1}{\sqrt{3}}$

Since $\cos x$ is positive, the terminal arm of angle x lies in Quadrant 1 or 4.

The reference angle is: $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = 54.7356\ldots°$

For $0 \leq x < 270°$, in Quadrant 1, $x \doteq 55°$

For $-90° \leq x < 0°$, in Quadrant 4, $x \doteq -55°$

To the nearest degree, the roots are: $x = 55°$ and $x = -55°$

The period of $\cos x$ is 360°, so the general solution is approximately: $x = \pm 55° + k360°$, $k \in \mathbb{Z}$

Check Your Understanding

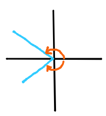
2. a) To the nearest hundredth, solve the equation $\cos x - 6 = 4\cos x - 4$ for ~~$-\frac{\pi}{2} \leq x < \frac{3\pi}{2}$~~ then write the general solution.

b) To the nearest degree, solve the equation $\sqrt{2}\csc x = -5$ for ~~$-180°$~~ $-\pi$ $\leq x <$ ~~$180°$~~ π, then write the general solution.

a) $\cos x - 6 = 4\cos x - 4$

$-3\cos x = 2$

$\cos x = -\frac{2}{3}$ ← cosine is negative in Q2 & Q3



$x = \cos^{-1}\left(-\frac{2}{3}\right)$

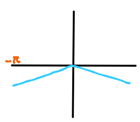
$x \doteq 2.30$ ← Q2

$2\pi - 2.30 \doteq 3.98$ ← Q3

$x = 2.30 + 2\pi n,\ 3.98 + 2\pi n$ where $n \in \mathbb{Z}$

b) $\csc x = -\frac{5}{\sqrt{2}}$

$\sin x = -\frac{\sqrt{2}}{5}$ ← sine is negative in Q3 & Q4



$x = \sin^{-1}\left(-\frac{\sqrt{2}}{5}\right)$

$x = -0.29$ ← Q4

$x = -\pi + 0.29 \doteq -2.85$ ← Q3

if $0 \leq x < 2\pi$

$-0.29 + 2\pi \doteq 6.00$ ← Q4

$\pi + 0.29 \doteq 3.42$ ← Q3

general solution:

$x = -0.29 + 2\pi n,\ -2.85 + 2\pi n$ where $n \in \mathbb{Z}$

Recall the strategy for solving a quadratic equation such as:

$5x^2 = 1 - 4x$	Move all the terms to one side.
$5x^2 + 4x - 1 = 0$	Factor.
$(5x - 1)(x + 1) = 0$	Equate each factor to 0, then solve for x.
Either $5x - 1 = 0$ or	$x + 1 = 0$
$5x = 1$	$x = -1$
$x = \frac{1}{5}$	

The algebraic strategies for solving a quadratic equation can be applied to solve a second-degree trigonometric equation. Consider solving the equation $5\cos^2 x = 1 - 4\cos x$ over the domain $0 \leq x < 2\pi$.

$5\cos^2 x = 1 - 4\cos x$	Move all the terms to one side.
$5\cos^2 x + 4\cos x - 1 = 0$	Factor the expression, treating $\cos x$ as the variable.
$(5\cos x - 1)(\cos x + 1) = 0$	Equate each factor to 0, then solve for $\cos x$.
Either $5\cos x - 1 = 0$ or	$\cos x + 1 = 0$
$5\cos x = 1$	$\cos x = -1$
$\cos x = \frac{1}{5}$	The terminal arm of angle x lies on the
$\cos x$ is positive when the terminal arm of angle x lies in Quadrant 1 or 4.	negative x-axis, so: $x = \pi$

In Quadrant 1, $x = \cos^{-1}\left(\frac{1}{5}\right)$

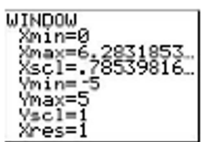
$x = 1.3694\ldots$

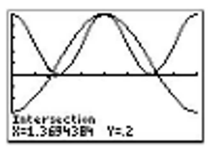
In Quadrant 4, $x = 2\pi - 1.3694\ldots$

$x = 4.9137\ldots$

The roots are: $x \doteq 1.37$, $x = \pi$, and $x \doteq 4.91$

The roots can be verified graphically.





The roots can also be verified by substituting in the original equation.

Example 3 **Solving a Second-Degree Trigonometric Equation by Using Square Roots**

Check Your Understanding

3. Use algebra to solve the equation $2\cos^2 x = 1$ over the domain $0° \leq x \leq 360°$.

$0 \quad 2\pi$

$2\cos^2 x = 1$

$\cos^2 x = \frac{1}{2}$

Use algebra to solve the equation $4\sin^2 x = 3$ over the domain $-360° \leq x \leq 0°$.

SOLUTION

$4\sin^2 x = 3$ Solve for $\sin x$.

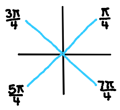
domain $0° \leq x \leq 360°$.

$0 \quad 2\pi$

$2\cos^2 x = 1$

$\cos^2 x = \frac{1}{2}$

$\cos x = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$



$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

SOLUTION

$4\sin^2 x = 3$ Solve for $\sin x$.

$\sin^2 x = \frac{3}{4}$

$\sin x = \pm\sqrt{\frac{3}{4}}$, or $\pm\frac{\sqrt{3}}{2}$

Since $\sin x$ is positive or negative, there is a solution in every quadrant.

The reference angle is: $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = 60°$

In Quadrant 1, $x = -360° + 60°$, or $-300°$
In Quadrant 2, $x = -180° - 60°$, or $-240°$
In Quadrant 3, $x = -180° + 60°$, or $-120°$
In Quadrant 4, $x = -60°$
Verify by substituting each root in the equation: $4\sin^2 x = 3$
The roots are: $x = -60°$, $x = -120°$, $x = -240°$, and $x = -300°$

When a second-degree equation cannot be solved by factoring or using square roots, the quadratic formula is used.

Example 4 Solving a Trigonometric Equation Using the Quadratic Formula

a) Use algebra to solve the equation $4\tan^2 x = 2\tan x + 1$ over the domain $-\pi \leq x \leq \pi$.
b) Determine the general solution.

Give the answers to the nearest hundredth.

SOLUTION

a) $4\tan^2 x = 2\tan x + 1$ Move all the terms to one side.
$4\tan^2 x - 2\tan x - 1 = 0$ This does not factor.

Use the quadratic formula: $\tan x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Substitute: $a = 4$, $b = -2$, $c = -1$

$$\tan x = \frac{2 \pm \sqrt{(-2)^2 - 4(4)(-1)}}{2(4)}$$

$$\tan x = \frac{2 \pm \sqrt{20}}{8}, \text{ or } \frac{1 \pm \sqrt{5}}{4}$$

Check Your Understanding

4. a) Use algebra to solve the equation $\cos x = 1 - 3\cos^2 x$ over the domain $-\pi \leq x \leq \pi$.

b) Determine the general solution.

Give the answers to the nearest hundredth.

$3\cos^2 x + \cos x - 1 = 0$

$\cos x = \frac{-1 \pm \sqrt{1^2 - 4(3)(-1)}}{2(3)}$

$= \frac{-1 \pm \sqrt{13}}{6}$

quadratic formula:
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Either $\tan x = \frac{1 + \sqrt{5}}{4}$ or $\tan x = \frac{1 - \sqrt{5}}{4}$

$\tan x$ is positive when the terminal arm of angle x lies in Quadrant 1 or 3.
The reference angle is:

$$\tan^{-1}\left(\frac{1 + \sqrt{5}}{4}\right) = 0.6802\ldots$$

In Quadrant 1, $x = 0.6802\ldots$
In Quadrant 3,
$x = -\pi + 0.6802\ldots$
$x = -2.4613\ldots$

$\tan x$ is negative when the terminal arm of angle x lies in Quadrant 2 or 4.
The reference angle is:

$$\tan^{-1}\left(\frac{\sqrt{5} - 1}{4}\right) = 0.2997\ldots$$

In Quadrant 2,
$x = \pi - 0.2997\ldots$
$x = 2.8418\ldots$
In Quadrant 4, $x = -0.2997\ldots$

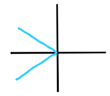
Verify by substituting each root in the given equation.
To the nearest hundredth, the roots are: $x = -2.46$, $x = -0.30$, $x = 0.68$, and $x = 2.84$

b) The period of $\tan x$ is π, so the general solution is approximately:
$x = 0.68 + \pi k, k \in \mathbb{Z}$ or $x = 2.84 + \pi k, k \in \mathbb{Z}$

$-\pi \le x \le \pi$

$\cos x = \frac{-1 \pm \sqrt{13}}{6}$

$x = \cos^{-1}\left(\frac{-1 + \sqrt{13}}{6}\right) = 2.45 \leftarrow Q2$



$x = -2.45 \leftarrow Q3$

$x = \cos^{-1}\left(\frac{-1 - \sqrt{13}}{6}\right) = 1.12 \leftarrow Q1$

$x = -1.12 \leftarrow Q4$

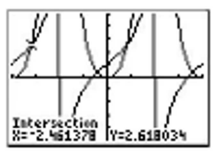
general solution:
$x = \pm 1.12 + 2\pi n$, $\pm 2.45 + 2\pi n$
where $n \in \mathbb{Z}$.

Try: p.592 #2,4,6,7,11,14

THINK FURTHER

In *Example 4*, when $\tan x$ is negative, why is the reference angle $\tan^{-1}\left(\frac{\sqrt{5} - 1}{4}\right)$?

The solution of the equation in *Example 4* can be verified by graphing.
On a graphing calculator, input $y = 4\tan^2 x$ and $y = 2\tan x + 1$, then determine the approximate x-coordinates of the points of intersection:



X = −2.461378
X = −0.2997086
X = 0.68021498
X = 2.8418841
These values match the values determined algebraically, so the solution is verified.

Discuss the Ideas

1. For equations of the form $\sin x = c$, $\cos x = c$, and $\tan x = c$, where c is a constant, what restrictions are there on the value of c for the equations to have real solutions?

2. How do you recognize when an exact solution to a trigonometric equation exists?

3. A second-degree trigonometric equation in terms of $\sin x$, $\cos x$, or $\tan x$ is solved over the domain $0 \leq x < 2\pi$. How many roots could the equation have?

Exercises

A

Use algebra to solve each equation. Give exact values when possible; otherwise write the roots to the nearest degree or the nearest hundredth of a radian. Verify the solutions.

4. Solve each equation over the domain $0 \leq x < 2\pi$.

a) $\sin x = \frac{\sqrt{3}}{2}$

b) $\tan x = \frac{1}{\sqrt{3}}$