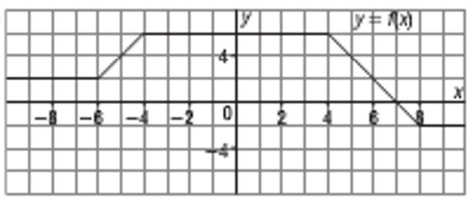
6.6 Combining Transformations of Sinusoidal Functions

FOCUS **Apply all transformations to graphs of sinusoidal functions.**

Get Started

The graph of $y = f(x)$ is shown.

On the same grid, sketch the graph of $y + 3 = \frac{1}{2}f(2(x - 4))$.



Construct Understanding

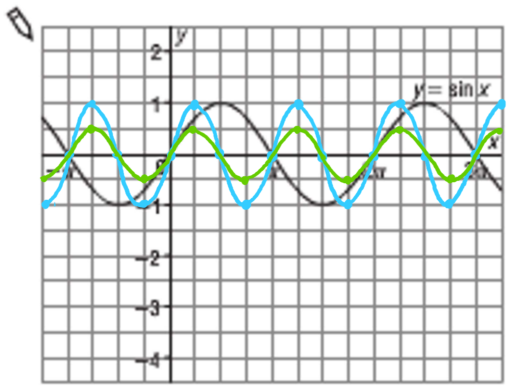
Construct Understanding

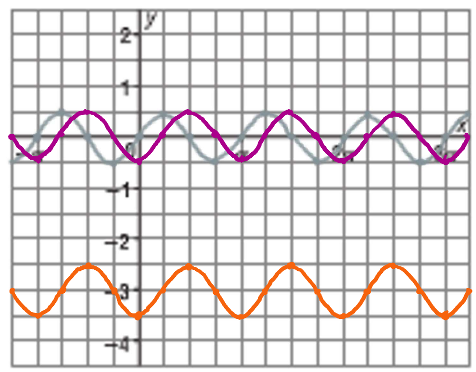
The graph of $y = \sin x$ is shown below.
On the same grid, sketch a graph of $y = \sin 2x$.
On the second grid, sketch graphs of:

$y = \frac{1}{2}\sin 2x \qquad y = \frac{1}{2}\sin 2\left(x - \frac{\pi}{4}\right)$

$y = \frac{1}{2}\sin 2\left(x - \frac{\pi}{4}\right) - 3$

Label each image graph with its equation.
Describe how the graph changes with each transformation.
Complete the table below.





	Equation of Function			
Characteristic	$y = \sin 2x$	$y = \frac{1}{2}\sin 2x$	$y = \frac{1}{2}\sin 2\left(x - \frac{\pi}{4}\right)$	$y = \frac{1}{2}\sin 2\left(x - \frac{\pi}{4}\right) - 3$
Period	$\frac{2\pi}{2} = \pi$			
Amplitude	1	$\frac{1}{2}$		
Domain of function	$x \in \mathbb{R}$			
Range of function	$-1 \le y \le 1$	$-\frac{1}{2} \le y \le \frac{1}{2}$		(subtract 3 from max/min) $-3.5 \le y \le -2.5$
Phase shift	none		$\frac{\pi}{4}$	
Zeros	$0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi \ldots$		(add $\pi/4$ to each zero) $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \ldots$	

The graph of a function $y = af(b(x - c)) + d$ is the image of the graph of $y = f(x)$ after transformations. The transformations depend on the values of the constants a, b, c, and d.

$y = af(b(x - c)) + d$				
Constant	***a***	***b***	***c***	***d***
Transformation applied to the graph of $y = f(x)$	vertical stretch or compression by a factor of $\lvert a \rvert$; if $a < 0$, there is also a reflection in the x-axis	horizontal stretch or compression by a factor of $\frac{1}{\lvert b \rvert}$; if $b < 0$, there is also a reflection in the y-axis	horizontal translation of c units	vertical translation of d units

In Lesson 6.5, these transformations were applied to the graphs of $y = \sin x$, $y = \cos x$, and $y = \tan x$. The transformations may change the period, the location of the centre line, any zeros, and the amplitude of a sinusoidal function. As a result, the range may also change.

The appearance of the graph of a trigonometric function can be predicted from its equation.

Example 1 Using Transformations to Sketch a Graph of a Trigonometric Function

Check Your Understanding

1. a) Predict how the graph of $y = \frac{1}{4}\cos 3\left(x + \frac{\pi}{6}\right) + 2$ is related to the graph of $y = \cos x$.

b) Sketch the graph of $y = \frac{1}{4}\cos 3\left(x + \frac{\pi}{6}\right) + 2$ for $-2\pi \leq x \leq 2\pi$, then list the characteristics of the function.

a) Predict how the graph of $y = 2\sin\frac{1}{2}\left(x - \frac{\pi}{3}\right) - 1$ is related to the graph of $y = \sin x$.

b) Sketch the graph of $y = 2\sin\frac{1}{2}\left(x - \frac{\pi}{3}\right) - 1$ for $0 \leq x \leq 4\pi$, then list the characteristics of the function.

SOLUTION

a) The graph of $y = 2\sin\frac{1}{2}\left(x - \frac{\pi}{3}\right) - 1$ is the image of the graph of $y = \sin x$ after the following transformations have been applied:

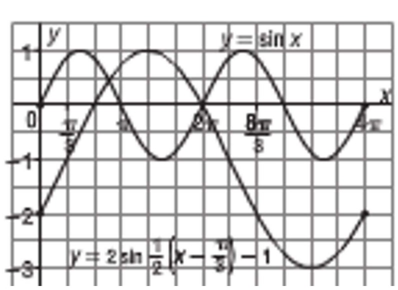
- a vertical stretch by a factor of 2
- a horizontal stretch by a factor of 2
- a horizontal translation (phase shift) of $\frac{\pi}{3}$ units right
- a vertical translation of 1 unit down

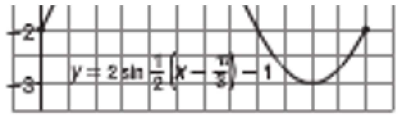
b) Sketch the graph of $y = \sin x$.

Since the phase shift is $\frac{\pi}{3}$, use a horizontal scale of 1 square represents $\frac{\pi}{3}$.

The horizontal stretch doubles the spacing between the zeros. The phase shift translates these points $\frac{\pi}{3}$ units right, then the vertical shift moves them 1 unit down. Plot these transformed points; they lie on the line $y = -1$, which is the centre line of the image graph. Choose other points on the graph of $y = \sin x$. For each point: double the x-coordinate; double the y-coordinate; shift the point $\frac{\pi}{3}$ units right; then 1 unit down

Plot the points on the grid and join them.

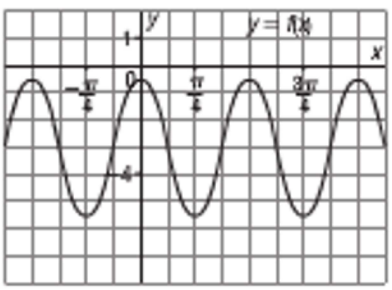




The amplitude is 2; the period is 4π; the phase shift is $\frac{\pi}{3}$; the zeros are $\frac{2\pi}{3}$ and 2π; the domain is $x \in \mathbb{R}$; and the range is $-3 \leq y \leq 1$.

Example 2 Writing the Equation of the Graph of a Trigonometric Function

Write an equation for the sinusoidal function graphed below, in terms of sin x.



SOLUTION

An equation has the form: $y = a \sin b(x - c) + d$

The equation of the centre line is $y = -3$, so the vertical translation is 3 units down and $d = -3$.

The amplitude is the distance between the centre line and a maximum or minimum point. This distance is: $a = \frac{5}{2}$

For the period, choose the x-coordinates of two adjacent minimum points, such as $-\frac{\pi}{4}$ and $\frac{\pi}{4}$. The period is: $\frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$

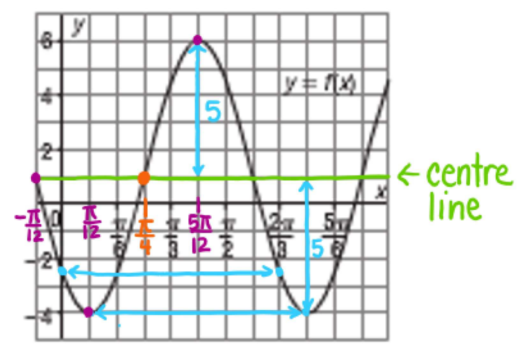
b is: $\dfrac{\text{period of } y = \sin x}{\text{period of given graph}} = \dfrac{2\pi}{\frac{\pi}{2}}$, or 4

Draw the line $y = -3$. Look for the closest point on either side of the y-axis where the sine function begins its cycle; that is, where the curve moves up to the right above the line. This point is at $x = -\frac{\pi}{8}$ on the given graph, so a possible phase shift is $-\frac{\pi}{8}$, and $c = -\frac{\pi}{8}$.

An equation is: $y = \frac{5}{2} \sin 4\left(x + \frac{\pi}{8}\right) - 3$

Check Your Understanding

2. Write an equation for the sinusoidal function graphed below, in terms of sin x.



amplitude: 5 → $a = 5$

$\frac{\text{max} - \text{min}}{2} = \frac{6 - (-4)}{2} = \frac{10}{2} = 5$

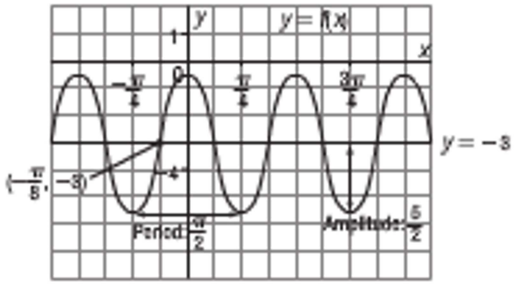
period: $\frac{2\pi}{3}$ → $b = 2\pi \div \text{period}$
$= 2\pi \div \frac{2\pi}{3}$
$= 3$

vertical translation: 1 unit up
(locate centre line) ↳ $d = 1$

phase shift:

- positive sine equation → look for a point on the centre line after which the graph goes up $(c = \frac{\pi}{4})$
  $y = 5\sin\left(3\left(x - \frac{\pi}{4}\right)\right) + 1$
- negative sine equation → look for a point on the centre line after which the graph goes down $(c = -\frac{\pi}{12})$
  $y = -5\sin\left(3\left(x + \frac{\pi}{12}\right)\right) + 1$

An equation is: $y = \frac{5}{2}\sin 4\left(x + \frac{\pi}{8}\right) - 3$



a point on the centre line after which the graph goes down ($c = -\frac{\pi}{12}$)

$y = -5\sin\left(3\left(x + \frac{\pi}{12}\right)\right) + 1$

- positive cosine equation → look for a maximum point ($c = \frac{5\pi}{12}$)

$y = 5\cos\left(3\left(x - \frac{5\pi}{12}\right)\right) + 1$

- negative cosine equation → look for a minimum point ($c = \frac{\pi}{12}$)

$y = -5\cos\left(3\left(x - \frac{\pi}{12}\right)\right) + 1$

p.535/536 #6(all) – write four equations for each graph (2 sine, 2 cosine / 2 pos., 2 neg.)

6a) $y = \frac{1}{2}\sin x$

$y = -\frac{1}{2}\sin(x - \pi)$

$y = \frac{1}{2}\cos\left(x - \frac{\pi}{2}\right)$

$y = -\frac{1}{2}\cos\left(x - \frac{3\pi}{2}\right)$

b) $y = \sin\left(x - \frac{3\pi}{2}\right) - 2$

$y = -\sin\left(x - \frac{\pi}{2}\right) - 2$

$y = \cos x - 2$

$y = -\cos(x - \pi) - 2$

c) $y = \sin\left(4\left(x + \frac{\pi}{8}\right)\right)$

$y = -\sin\left(4\left(x - \frac{\pi}{8}\right)\right)$

$y = \cos 4x$

$y = -\cos\left(4\left(x - \frac{\pi}{4}\right)\right)$

d) $y = \sin\left(x + \frac{\pi}{6}\right)$

$y = -\sin\left(x - \frac{5\pi}{6}\right)$

$y = \cos\left(x - \frac{\pi}{3}\right)$

$y = -\cos\left(x - \frac{4\pi}{3}\right)$