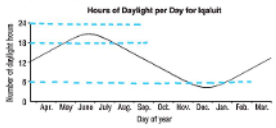
6.5 Trig Functions

6.5 Trigonometric Functions

FOCUS **Define the trigonometric functions and identify single transformations.**

Get Started

This graph shows how the number of hours of daylight in Iqaluit varies throughout the year.



Approximately how many hours of daylight are there on the longest day of the year?

21 hours

Approximately how many hours of daylight are there on the shortest day of the year?

4 hours

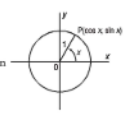
Why is it reasonable to expect this pattern to repeat annually?

annual orbit around the sun

Construct Understanding

Use graphing technology.
Graph the function $y = a \sin x$ for different integer values of a.
How does the graph of $y = a \sin x$ change as the value of a changes?
What remains the same?

Point $P(\cos x, \sin x)$ lies on the unit circle. OP is the terminal arm of an angle, x radians, in standard position.



For a central angle in the unit circle, the radian measure of the angle is the length of the arc that subtends the angle, which is a real number. So, radians can be used to define **trigonometric functions** of a real number.

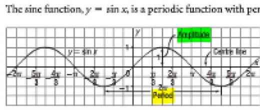
Trigonometric Functions

For any real number x:

- $y = \sin x$ is the value of the sine ratio for an angle measuring x radians
- $y = \cos x$ is the value of the cosine ratio for an angle measuring x radians
- $y = \tan x$ is the value of the tangent ratio for an angle measuring x radians

A function that repeats its values in regular intervals over its domain is a **periodic function**. The length of each interval, or *cycle*, measured along the horizontal axis is called the **period** of the function.

The sine function, $y = \sin x$, is a periodic function with period 2π.

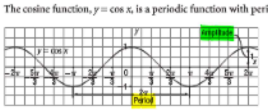


The domain of the sine function is: $x \in \mathbb{R}$

The zeros are: $0, \pm\pi, \pm 2\pi, \ldots$; that is, the zeros have the form $k\pi, k \in \mathbb{Z}$

The function has a maximum value of 1 and a minimum value of -1. So, the range is: $-1 \leq y \leq 1$

The cosine function, $y = \cos x$, is a periodic function with period 2π.



The domain of the cosine function is: $x \in \mathbb{R}$

The zeros are: $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$; that is, the zeros have the form $(2k + 1)\frac{\pi}{2}, k \in \mathbb{Z}$

The function has a maximum value of 1 and a minimum value of -1. So, the range is: $-1 \leq y \leq 1$

Functions whose graphs have the same shape as $y = \sin x$ or $y = \cos x$ are **sinusoidal functions**. A sinusoidal function has a maximum value and a minimum value that are equidistant from the *centre line* of the graph; this is the horizontal line that is halfway between the maximum

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Functions whose graphs have the same shape as $y = \sin x$ or $y = \cos x$ are **sinusoidal functions**. A sinusoidal function has a maximum value and a minimum value that are equidistant from the *centre line* of the graph; this is the horizontal line that is halfway between the maximum points and the minimum points. The **amplitude** of a sinusoidal function is the distance of a maximum or minimum point from the centre line.

THINK FURTHER

How can you use the position of the centre line and the amplitude to determine the maximum and minimum values of a sinusoidal function and its range?

Transformations can be applied to the graph of a trigonometric function:

- horizontal stretches or compressions
- vertical stretches or compressions
- reflections in horizontal and vertical axes
- horizontal translations
- vertical translations

Example 1 Determining the Amplitude of a Trigonometric Function

Check Your Understanding

1. Determine the amplitude of the graph of each function.
   a) $y = \frac{2}{3}\sin x$
   b) $y = -4\cos x$

amplitude = vertical stretch/compression
*always positive

a) $\frac{2}{3}$ b) 4

Determine the amplitude of the graph of each function.

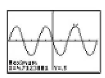
a) $y = 3\cos x$ **b)** $y = -\frac{1}{2}\sin x$

SOLUTION

a) The graph of $y = 3\cos x$ is the image after the graph of $y = \cos x$ has been stretched vertically by a factor of 3.
The amplitude of $y = \cos x$ is 1.
So, the amplitude of $y = 3\cos x$ is 3.

b) The graph of $y = -\frac{1}{2}\sin x$ is the image after the graph of $y = \sin x$ has been compressed vertically by a factor of $\frac{1}{2}$ and reflected in the x-axis.
The amplitude of $y = \sin x$ is 1.
So, the amplitude of $y = -\frac{1}{2}\sin x$ is $\frac{1}{2}$.

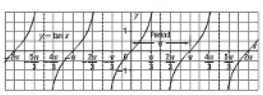
Graphing technology can be used to verify the amplitudes in *Example 1*.
For part b, $y = -\frac{1}{2}\sin x$:
The y-coordinate of a maximum point is 0.5.



THINK FURTHER

Use the unit circle to explain why the period of the graph of $y = \tan x$ is π.

The tangent function, $y = \tan x$, is a periodic function with period π.



The graph has asymptotes with equations $x = \pm\frac{\pi}{2}, x = \pm\frac{3\pi}{2}, \ldots$; so the domain of the tangent function is: $x \neq \frac{n\pi}{2}$, where n is an odd integer
The zeros are: $0, \pm\pi, \pm 2\pi, \ldots$; that is, the zeros have the form $k\pi, k \in \mathbb{Z}$
The function has no maximum or minimum values, so its graph has no amplitude. The range of the function is: $y \in \mathbb{R}$

Example 2 Determining the Period of a Trigonometric Function

Check Your Understanding

2. Determine the period of each function.
   a) $y = \cos 6x$ b) $y = \tan\frac{2}{3}x$
   c) $y = \sin\frac{x}{7}$

divide the period by the coefficient of x
period of sinx is 2π
of cosx is 2π
of tanx is π

a) $\frac{2\pi}{6} = \frac{\pi}{3}$
b) $\frac{\pi}{2/3} = \frac{3\pi}{2}$
c) $\frac{2\pi}{1/7} = 14\pi$

Determine the period of each function.

a) $y = \sin 2x$ **b)** $y = \cos\frac{3}{4}x$ **c)** $y = \tan\frac{x}{6}$

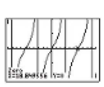
SOLUTION

a) The graph of $y = \sin 2x$ is the image after the graph of $y = \sin x$ has been compressed horizontally by a factor of $\frac{1}{2}$.
So, the period of $y = \sin 2x$ is $\frac{1}{2}$ the period of $y = \sin x$.
The period of $y = \sin 2x$ is: $\frac{1}{2}(2\pi) = \pi$

b) The graph of $y = \cos\frac{3}{4}x$ is the image after the graph of $y = \cos x$ has been stretched horizontally by a factor of $\frac{4}{3}$.
So, the period of $y = \cos\frac{3}{4}x$ is $\frac{4}{3}$ the period of $y = \cos x$.
The period of $y = \cos\frac{3}{4}x$ is: $\frac{4}{3}(2\pi) = \frac{8}{3}\pi$

c) The graph of $y = \tan\frac{x}{6}$ is the image after the graph of $y = \tan x$ has been stretched horizontally by a factor of 6.
So, the period of $y = \tan\frac{x}{6}$ is 6 times the period of $y = \tan x$.
The period of $y = \tan\frac{x}{6}$ is: 6π

Graphing technology can be used to check the periods in *Example 2*.
For part c, $y = \tan\frac{x}{6}$:
For the tangent function, adjacent zeros indicate the period.
One zero is 0, the next zero is 6π, as shown.



The results of *Example 2* can be generalized.

Period of a Trigonometric Function

One zero is 0, the next zero is 6π, as shown.

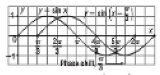
The results of *Example 2* can be generalized.

Period of a Trigonometric Function

The period of $y = \sin bx$ and $y = \cos bx$ is: $\frac{2\pi}{b}$, $b > 0$

The period of $y = \tan bx$ is: $\frac{\pi}{b}$, $b > 0$

Consider the functions $y = \sin x$ and $y = \sin\left(x - \frac{\pi}{3}\right)$.



The graph of $y = \sin\left(x - \frac{\pi}{3}\right)$ is congruent to the graph of $y = \sin x$, but it has been translated $\frac{\pi}{3}$ units right.

In general, the graph of $y = \sin(x - c)$ is the image after the graph of $y = \sin x$ has been translated c units horizontally; this distance is the **phase shift** of the function.

Example 3 Determining the Phase Shift of a Trigonometric Function

a) Determine the phase shift of the function $y = \tan\left(x + \frac{\pi}{2}\right)$.

b) Sketch graphs of $y = \tan x$ and $y = \tan\left(x + \frac{\pi}{2}\right)$ for $0 \le x \le 2\pi$.

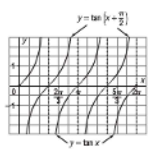
SOLUTION

a) Compare $y = \tan\left(x + \frac{\pi}{2}\right)$

with $y = \tan(x - c)$

The phase shift is: $-\frac{\pi}{2}$ ← phase shift is negative if graph moves left

b) Graph $y = \tan x$, then translate the graph $\frac{\pi}{2}$ units left to obtain the graph of $y = \tan\left(x + \frac{\pi}{2}\right)$.



Check Your Understanding

3. a) Determine the phase shift of the function $y = \cos\left(x - \frac{\pi}{6}\right)$.

b) Sketch graphs of $y = \cos x$ and $y = \cos\left(x - \frac{\pi}{6}\right)$ for $0 \le x \le 2\pi$.

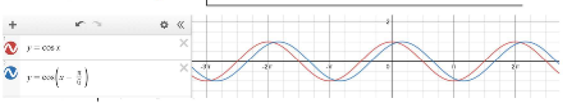
$y = f(x-1)$

→ translation 1 unit right

$y = \cos\left(x - \frac{\pi}{6}\right)$

→ translation $\frac{\pi}{6}$ units right

⇒ phase shift : $\frac{\pi}{6}$



Example 4 Identifying Transformations of the Graph of a Sinusoidal Function

Describe how the graph of each function below relates to the graph of $y = \cos x$. Then, on the same grid, sketch the graphs of $y = \cos x$ and each function below, for $0 \le x \le 2\pi$.

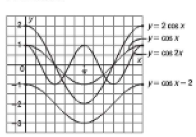
a) $y = 2\cos x$ b) $y = \cos 2x$ c) $y = \cos x - 2$

SOLUTION

a) The graph of $y = 2\cos x$ is the image after the graph of $y = \cos x$ has been stretched vertically by a factor of 2.
To graph $y = 2\cos x$: for points on the graph of $y = \cos x$, multiply each y-coordinate by 2.

b) The graph of $y = \cos 2x$ is the image after the graph of $y = \cos x$ has been compressed horizontally by a factor of $\frac{1}{2}$.
To graph $y = \cos 2x$: for points on the graph of $y = \cos x$, divide each x-coordinate by 2.

c) The graph of $y = \cos x - 2$ is the image after the graph of $y = \cos x$ has been translated 2 units down.
To graph $y = \cos x - 2$, move each point on $y = \cos x$ 2 units down.



Check Your Understanding

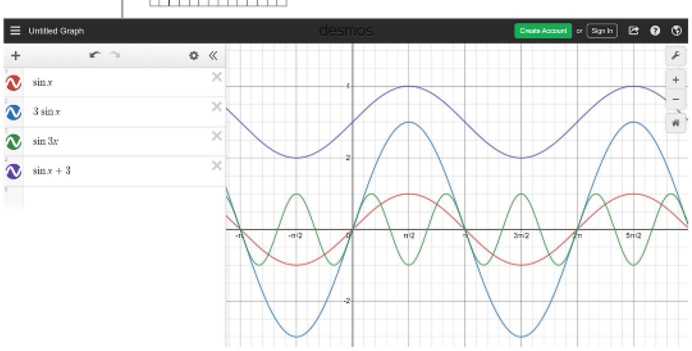
4. Describe how the graph of each function below relates to the graph of $y = \sin x$. Then, on the same grid, sketch the graphs of $y = \sin x$ and each function below, for $0 \le x \le 2\pi$. amplitude = 3

a) $y = 3\sin x$ ← vertical stretch

b) $y = \sin 3x$ ← horizontal compression period = $\frac{2\pi}{3}$

c) $y = \sin x + 3$

↑ translation 3 units up



p.521 : #3-5,7,9

p.521: #3-5, 7, 9