6.2 Geometric Series

A geometric series is the sum of the terms of a geometric sequence. For example, a geometric sequence is 6, 12, 24, 48, … The related geometric series is 6 + 12 + 24 + 48 + …

Find the sum of the first 5 terms of the geometric series 1 + 3 + 9 + …

$1 + 3 + 9 + 27 + 81 = 121$

We use the notation S_n to represent the sum of the first n terms of a series. Using some fancy math, we can derive an equation to make the computation of a series with many terms easier. Multiplying the sum by r and then subtracting this from the original equation yields the following:

$$\begin{aligned} S_n &= a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} \\ -\ (S_n r &= \quad ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n) \end{aligned}$$

$$S_n - S_n r = a - ar^n$$
$$S_n(1 - r) = a(1 - r^n)$$

$$\boxed{S_n = \frac{a(1-r^n)}{1-r}, r \neq 1}$$

Example 1: Determine the sum of the first 10 terms of the geometric series:

a) 4 + 12 + 36 + …

$a = 4; r = 3; n = 10$

$S_{10} = \frac{4(1-3^{10})}{1-3}$

$= 118096$

b) 6 + 3 + 1.5 + …

$a = 6; r = 0.5; n = 10$

$S_{10} = \frac{6(1-0.5^{10})}{1-0.5}$

$= 11.98828\ldots$

c) 6 – 3 + 1.5 - …

$a = 6; r = -0.5; n = 10$

$S_{10} = \frac{6(1-(-0.5)^{10})}{1-(-0.5)}$

$= \frac{6(1-0.5^{10})}{1.5}$

$= 3.996\ldots$

Example 2: The sum of the first 14 terms of a geometric series is 16 383. The common ratio is –2. Determine the first term.

$S_{14} = 16383; n = 14; r = -2$

$\frac{a(1-(-2)^{14})}{1-(-2)} = 16383$

$\frac{a(1-2^{14})}{3} = 16383$

$a = \frac{16383 \cdot 3}{1-2^{14}} = -3$

The first term is -3.

$$a = \frac{\qquad}{1-2^{14}} = -3$$

Example 3: Determine the sum of terms 9 to 15 in the geometric series 2 – 6 + 18 – ... $a=2;\ r=-3$

$$S_{15} - S_8 = \frac{2(1-(-3)^{15})}{1-(-3)} - \frac{2(1-(-3)^8)}{1-(-3)}$$
$$= \frac{2(1+3^{15})}{4} - \frac{2(1-3^8)}{4}$$
$$= 7177734$$

Example 4: Determine the sum the geometric series 2 – 6 + 18 – ... + 1458.

① Figure out which term is 1458.

$t_n = ar^{n-1}$
$1458 = 2(-3)^{n-1}$
$729 = (-3)^{n-1}$
$(-3)^6 = (-3)^{n-1}$
$\therefore 6 = n-1 \Rightarrow n=7 \Rightarrow t_7 = 1458$

② $S_7 = \frac{2(1-(-3)^7)}{1-(-3)}$
$= 1094$

The sum is 1094.

Sigma Notation

Sigma notation allows us to represent a geometric series in a more compact way. Consider the following series:

$$20 + 40 + 80 + 160 = \sum_{k=1}^{4} 20(2)^{k-1}$$

(4 ← final value; Σ = "the sum of"; $k=1$ ← starting value)

Example 5: For the geometric series: $\sum_{k=1}^{n} -3(5)^{k-1}$

a) Write the first 4 terms of the series.

$t_1 = -3(5)^{1-1} = -3(5)^0 = -3$
$t_2 = -3(5)^{2-1} = -3(5)^1 = -15$
$t_3 = -3(5)^{3-1} = -3(5)^2 = -75$
$t_4 = -3(5)^{4-1} = -3(5)^3 = -375$

The terms are $-3, -15, -75, -375$.

b) Determine the sum of the series when its last term –46875.

$-3(5)^{k-1} = -46875$
$5^{k-1} = 15625$
$5^{k-1} = 5^6$
$\therefore k-1 = 6$
$k = 7$

$S_7 = \frac{-3(1-5^7)}{1-5}$
$= -58593$

The sum is -58593.

Assignment: handout

6.2 Assignment Name: _____

1. Why do the terms in some geometric series alternate between positive and negative numbers?

2. Use the given data about each geometric series to determine the indicated value.

a) $a = 1, r = 0.3$; determine S_8

b) $a = \frac{3}{4}, r = \frac{1}{2}$; determine S_4

3. Determine S_6 for each geometric series.

a) $2 + 10 + 50 + \cdots$

b) $80 - 40 + 20 - \cdots$

c) $\sum_{k=1}^{n} 3(2)^{k-1}$

d) $\sum_{k=1}^{n} 2(-3)^{k}$

4. For each geometric series, determine how many terms it has then calculate its sum.

a) $1 - 2 + 4 - 8 + \ldots - 512$

b) $-6561 + 2187 - 729 + 243 - \ldots - 1$

5. On Monday, a person's post on social media is reposted by 3 people. On Tuesday, each of the 3 reposts is again reposted by 3 different people. On Wednesday, each of the now 9 reposts is again reposted by 3 different people.

a) Write the total number of reposts as a geometric series. (Do not include the original post.) Represent the series with sigma notation. What is the first term? What is the common ratio?

b) Suppose this pattern continued for 1 week. What is the total number of reposts?

6. Determine S_{11} for the geometric series $\sum_{k=1}^{10} a(-2)^{k-1} = 1705$.

Answers:

2. a) approximately 1.428 **b)** approximately 1.406 **3. a)** 7812 **b)** 52.5 **c)** 189 **d)** 1092
4. a) 10 terms; –341 **b)** 9 terms; –4921 **5. a)** 3 + 9 + 27; 3; 3 **b)** 3279 reposts **6.** –3415