5.8 Solving Problems with Exponents and Logarithms

FOCUS Use exponents and logarithms to model and solve problems.

Get Started

Use the formula for the sum of a geometric series to determine the sum of this series: $4 + 4.4 + 4.84 + \dots + 7.086244$

common ratio: $\frac{4.4}{4} = 1.1$

$t_n = ar^{n-1}$

$\uparrow t_7$

$4(1.1)^{n-1} = 7.086244$

$\div 4 \qquad \div 4$

$\log 1.1^{n-1} = \log 1.771561$

$(n-1)\log 1.1 = \log 1.771561$

$n-1 = \frac{\log 1.771561}{\log 1.1}$

$n-1 = 6 \quad \therefore n = 7$

$S_7 = \frac{4(1-1.1^7)}{1-1.1}$

$= 37.948684$

Construct Understanding

A principal of $5000 is invested at 3% annual interest, compounded monthly. Use algebra to determine the time, to the nearest year, it will take for the investment to double.
Use graphing technology to verify the answer.

$10000 = 5000\left(1 + \frac{0.03}{12}\right)^{12t}$

$2 = (1.0025)^{12t}$

$\log 2 = \log(1.0025)^{12t}$

$\log 2 = 12t \log 1.0025$

$\frac{\log 2}{12 \log 1.0025} = t$

$23.133775\ldots = t$

It will take about 23 years.

Try p.435 #3

THINK FURTHER

For the problem on page 430, how would the solution change if the interest was compounded semi-annually instead of monthly?

When a series of equal investments is made at equal time intervals, and the compounding period for the interest is equal to the time interval for the investments, the amount in dollars, or *future value FV*, of these investments can be determined using this formula:

$$FV = \frac{R[(1+i)^n - 1]}{i}$$

where R dollars is the regular investment,
i is the interest rate per compounding period, and
n is the number of investments

Example 1 Solving a Problem Involving Future Value

Determine how many monthly investments of $100 would have to be made into a savings account that pays 6% annual interest, compounded monthly, for the future value to be $100 000.

SOLUTION

Use: $FV = \frac{R[(1+i)^n - 1]}{i}$, where n is the number of monthly investments

Substitute: $FV = 100\,000$; $R = 100$; $i = \frac{0.06}{12}$, or 0.005

$100\,000 = \frac{100[(1+0.005)^n - 1]}{0.005}$ Simplify.

$100\,000 = 20\,000(1.005^n - 1)$ Divide each side by 20 000.

$5 = 1.005^n - 1$

$6 = 1.005^n$ Take the common logarithm of each side.

$\log 6 = \log 1.005^n$ Apply the power law.

$\log 6 = n \log 1.005$ Solve for n.

$n = \frac{\log 6}{\log 1.005}$

$n = 359.2470\ldots$

The number of investments is 360.

Check Your Understanding

1. Determine how many monthly investments of $200 would have to be made into an account that pays 6% annual interest, compounded monthly, for the future value to be $100 000.

$FV = \frac{R[(1+i)^n - 1]}{i}$

$i = \frac{0.06}{12} = 0.005$

$100000 = \frac{200[1.005^n - 1]}{0.005}$

$2.5 = 1.005^n - 1$

$\log 3.5 = \log 1.005^n$

$\log 3.5 = n \log 1.005$

$\frac{\log 3.5}{\log 1.005} = n$

$n = 251.178\ldots$

252 investments must be made.

THINK FURTHER

In *Example 1*, suppose 180 monthly investments of $100 were made into the account. Would the future value be $50 000? Justify the answer.

Many people borrow money to finance a purchase. A loan is usually repaid by making regular equal payments for a fixed period of time. The amount borrowed is called the *present value, PV,* of the loan. The following formula relates the present value to n equal payments of R dollars each, when the interest rate per compounding period is i. The compounding period is equal to the time between payments. The first payment is made after a time equal to the compounding period.

$$PV = \frac{R[1 - (1 + i)^{-n}]}{i}$$

Example 2 Solving a Problem Involving Loans

A person wants to borrow $200 000 as a mortgage to buy a house. The person can afford to pay $1500 a month. The mortgage will be repaid with equal monthly payments at 4% annual interest, compounded monthly. How many monthly payments will the person make?

SOLUTION

Use the formula:

$$PV = \frac{R[1 - (1 + i)^{-n}]}{i} \quad \text{Substitute: } PV = 200\,000, R = 1500, i = \frac{0.04}{12}$$

$$200\,000 = \frac{1500\left[1 - \left(1 + \frac{0.04}{12}\right)^{-n}\right]}{\frac{0.04}{12}} \quad \text{Simplify.}$$

$$\frac{200\,000\left(\frac{0.04}{12}\right)}{1500} = 1 - \left(1 + \frac{0.04}{12}\right)^{-n}$$

$$\frac{4}{9} = 1 - \left(1 + \frac{0.04}{12}\right)^{-n}$$

$$\left(1 + \frac{0.04}{12}\right)^{-n} = \frac{5}{9} \quad \text{Take the logarithm of each side.}$$

Check Your Understanding

2. A person borrows $15 000 to buy a car. The person can afford to pay $300 a month. The loan will be repaid with equal monthly payments at 6% annual interest, compounded monthly. How many monthly payments will the person make?

$$PV = \frac{R[1-(1+i)^{-n}]}{i}$$

$$i = \frac{0.06}{12} = 0.005$$

$$15000 = \frac{300[1-1.005^{-n}]}{0.005}$$

$$0.25 = 1 - 1.005^{-n}$$

$$1.005^{-n} = 1 - 0.25$$

$$\log 1.005^{-n} = \log 0.75$$

$$-n \log 1.005 = \log 0.75$$

$$-n = \frac{\log 0.75}{\log 1.005}$$

$$-n \doteq -57.68$$

$$n \doteq 57.68$$

58 payments will be required.
(The last payment will be less than $300.)

Try p.436 # 6ab, 7ab

$$\log\left(1+\frac{0.04}{12}\right)^{-n} = \log\left(\frac{5}{9}\right)$$

$$-n\log\left(1+\frac{0.04}{12}\right) = \log\left(\frac{5}{9}\right)$$

$$n = \frac{\log\left(\frac{5}{9}\right)}{-\log\left(1+\frac{0.04}{12}\right)}$$

$$n = 176.6297\ldots$$

There will be 177 monthly payments; the last payment will be less than the others.

When physical quantities have a large range of values, they are measured using a *logarithmic scale*. Some examples include the Richter scale, the decibel scale, and the pH scale.

In 1935, Charles Richter defined the magnitude, M, of an earthquake to be:

$$M = \log\left(\frac{I}{S}\right)$$

The intensity of the vibrations of an earthquake, I microns, is measured on a seismograph that is 100 km away from the *epicentre* of the earthquake. This intensity is compared to the intensity, S, of a standard earthquake, which has a seismograph reading of 1 micron and can barely be detected. The logarithmic scale for measuring the intensity of earthquakes is the **Richter scale.** Each increase of 1 unit on the logarithmic scale represents a 10-fold increase in intensity. For example, an earthquake of magnitude 9.0 in Japan in March 2011 was 100 times as intense as an earthquake of magnitude 7.0 in the same country in July 2011.

Example 3 Solving a Problem Involving the Richter Scale

In June 2010, Ontario and Quebec experienced an earthquake with magnitude 5.0. In January of the same year, Haiti experienced an earthquake with magnitude 7.0.

a) Calculate the intensity of the Haiti earthquake in terms of a standard earthquake.

b) Calculate the intensity of the Ontario-Quebec earthquake in terms of a standard earthquake.

c) How many times as intense as the Ontario-Quebec earthquake was the Haiti earthquake?

Check Your Understanding

3. The most intense earthquake ever recorded was in Chile in May 1960, with a magnitude of 9.5.

a) Calculate the intensity of the earthquake in Chile in terms of a standard earthquake.

b) How many times as intense as the Haiti earthquake was the Chile earthquake? Give the answer to the nearest whole number.

a) $M = \log\left(\frac{I}{S}\right)$

$9.5 = \log\left(\frac{I}{S}\right)$

$\frac{I}{S} = 10^{9.5}$

$I = 10^{9.5}S$

The earthquake was $10^{9.5}$ times as intense as a standard earthquake.

b)

$$\frac{\text{intensity of Chile earthquake}}{\text{intensity of Haiti earthquake}}$$

$$= \frac{10^{9.5}S}{10^{7}S}$$

$$= 10^{9.5-7}$$

$$= 10^{2.5}$$

$$\doteq 316$$

The earthquake in Chile was about 316 times as intense as the earthquake in Haiti.

Assignment:
p.435 #3,4,6,7,9-11

SOLUTION

a) Use: $M = \log\left(\frac{I}{S}\right)$ Substitute: $M = 7$

$7 = \log\left(\frac{I}{S}\right)$ To solve for I, write an exponential equation.

$\frac{I}{S} = 10^7$

$I = 10^7 S$

The earthquake was 10^7 times as intense as a standard earthquake.

b) Use: $M = \log\left(\frac{I}{S}\right)$ Substitute: $M = 5$

$5 = \log\left(\frac{I}{S}\right)$

$\frac{I}{S} = 10^5$

$I = 10^5 S$

The earthquake was 10^5 times as intense as a standard earthquake.

c) To compare the two earthquakes, divide their intensities.

$$\frac{\text{the intensity of the Haiti earthquake}}{\text{the intensity of the Ontario-Quebec earthquake}} = \frac{10^7 S}{10^5 S}$$
$$= 10^2\text{, or } 100$$

The earthquake in Haiti was 100 times as intense as the Ontario-Quebec earthquake.

Example 3 illustrates that an increase of 2 units on the Richter scale represents an earthquake intensity that is 100 times as great.

Discuss the Ideas

1. How are the formulas for present value and future value alike? How are they different?

2. Why is the Richter scale called a *logarithmic scale*? Why are the intensities of two earthquakes compared instead of calculated?