5.7 Solving Logarithmic and Exponential Equations

FOCUS Solve logarithmic and exponential equations algebraically.

Get Started

Simplify each expression.

$\log_4 100 + \log_4 2$

$= \log_4 200$

$\log_5 35 - \log_5 7$

$= \log_5\left(\frac{35}{7}\right)$

$= \log_5 5$

$= 1$

Construct Understanding

Use algebra to solve this equation: $\log_2 x + \log_2 x = 2$

Verify the solution.

$\log_2 x + \log_2 x = 2$

$2\log_2 x = 2$

$\div 2 \quad \div 2$

$\log_2 x = 1$

$2^1 = x$

$x = 2$

verify: $LS = \log_2 2 + \log_2 2$

$= 1 + 1$

$= 2$

$= RS$

A **logarithmic equation** is an equation that contains the logarithm of a variable.
The laws of logarithms may be used to solve logarithmic equations.

Check Your Understanding

1. Solve: $\log_3 9x + \log_3 x = 4$
   Verify the solution.

$\log_3 9x^2 = 4$

$3^4 = 9x^2$

$\div 9 \quad \div 9$

$9 = x^2$

$\pm\sqrt{9} = x$

$\pm 3 = x$

since $x > 0$, $\boxed{x = 3}$

verify:

$LS = \log_3 9(3) + \log_3 3$

$= \log_3 27 + \log_3 3$

$= 3 + 1$

$= 4$

$= RS$

Example 1 Solving a Logarithmic Equation Involving $\log_b dx$

Solve: $5 = \log_2 x + \log_2 2x$
Verify the solution.

SOLUTION

$5 = \log_2 x + \log_2 2x$ Use the product law of logarithms.

$5 = \log_2(x)(2x)$

$5 = \log_2 2x^2$ Write as an exponential statement.

$2x^2 = 2^5$

$x^2 = 2^4$

$x^2 = 16$

$x = \pm\sqrt{16}$

$x = \pm 4$

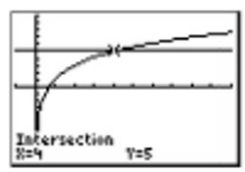
A logarithm is not defined for a negative number, so the solution is $x = 4$.

To verify $x = 4$, substitute in the original equation.

$$\begin{aligned} \text{R.S.} &= \log_2 x + \log_2 2x \\ &= \log_2 4 + \log_2 8 \\ &= 2 + 3 \\ &= 5 \\ &= \text{L.S.} \end{aligned}$$

Since the left side is equal to the right side, the solution is verified.

The solution of *Example 1* can be verified by solving the equation $5 = \log_2 x + \log_2 2x$ graphically. Use the change of base formula. In a graphing calculator, input:



$$y = 5 \text{ and } y = \frac{\log(x)}{\log(2)} + \frac{\log(2x)}{\log(2)}$$

Use the intersect feature from the CALC menu to determine the x-coordinate of the point of intersection. The solution is $x = 4$.

In *Example 1*, $x = -4$ is an extraneous solution.
A logarithm is defined for only positive numbers so, in the equation $5 = \log_2 x + \log_2 2x$, $x > 0$. However, when the laws of logarithms were used to combine the logarithms, the term $\log_2 2x^2$ was obtained. Since x^2 is positive for all values of x except $x = 0$, this new equation is defined for all real numbers except $x = 0$. Therefore $x = -4$ is a solution of the equation $5 = \log_2 2x^2$, but not of the original equation $5 = \log_2 x + \log_2 2x$.

Example 2 — Solving a Logarithmic Equation Involving $\log_b(dx - a)$

Solve, then verify each equation.

a) $2 \log x - \log (x + 2) = \log (2x - 3)$

b) $\log_6(x + 3) + \log_6(x + 4) = 1$

SOLUTION

a) $2 \log x - \log (x + 2) = \log (2x - 3)$

Consider the values of x for which each logarithm is defined.

$x > 0 \qquad x + 2 > 0 \qquad 2x - 3 > 0$

$\qquad\qquad x > -2 \qquad\qquad x > 1.5$

So, $x > 1.5$

$2 \log x - \log (x + 2) = \log (2x - 3)$ — Use the laws of logarithms.

$\log x^2 - \log (x + 2) = \log (2x - 3)$

$$\log\left(\frac{x^2}{x + 2}\right) = \log (2x - 3)$$

Write both sides as exponents of 10.

$\frac{x^2}{x + 2} = 2x - 3$ — Multiply each side by $x + 2$.

$x^2 = (2x - 3)(x + 2)$

$x^2 = 2x^2 + x - 6$

$x^2 + x - 6 = 0$ — Solve by factoring.

$(x + 3)(x - 2) = 0$

$x + 3 = 0 \quad \text{or} \quad x - 2 = 0$

$x = -3 \qquad\qquad x = 2$

Since $x > 1.5$, $x = -3$ is non-permissible, so it is an extraneous root.

Substitute $x = 2$ in the original equation to verify this solution.

L.S.	R.S.
$= 2 \log x - \log (x + 2)$	$= \log (2x - 3)$
$= 2 \log 2 - \log (2 + 2)$	$= \log (2(2) - 3)$
$= \log 2^2 - \log 4$	$= \log 1$
$= \log 4 - \log 4$	$= 0$
$= 0$	

Since the left side is equal to the right side, the solution is verified.

b) $\log_6(x + 3) + \log_6(x + 4) = 1$

Consider the values of x for which each logarithm is defined.

$x + 3 > 0 \qquad x + 4 > 0$

$x > -3 \qquad\qquad x > -4$

So, $x > -3$

$\log_6(x + 3) + \log_6(x + 4) = 1$ — Use the product law.

$\log_6[(x + 3)(x + 4)] = 1$

$\log_6(x^2 + 7x + 12) = 1$ — Write in exponential form.

$x^2 + 7x + 12 = 6^1$

$x^2 + 7x + 6 = 0$ — Factor.

$(x + 6)(x + 1) = 0$

$x + 6 = 0 \quad \text{or} \quad x + 1 = 0$

$x = -6 \qquad\qquad x = -1$

Check Your Understanding

2. Solve, then verify each equation.

a) $\log 6x = \log (x + 6) + \log (x - 1)$

b) $3 = \log_2(x + 2) + \log_2 x$

a) $\log 6x \quad \log(x+6) \quad \log(x-1)$

$x > 0 \quad x > -6 \quad x > 1$

$\therefore x > 1$

$\log 6x = \log (x+6)(x-1)$

$6x = (x+6)(x-1)$

$6x = x^2 + 5x - 6$

$0 = x^2 - x - 6$

$0 = (x-3)(x+2)$

$\boxed{x = 3}$ or $x = -2$

extraneous root

verify:

$LS = \log 6(3) \qquad RS = \log (3+6) + \log (3-1)$

$= \log 18 \qquad = \log 9 + \log 2$

$= \log 18$

$= LS$

b) $\log_2(x+2) \quad \log_2 x$

$x > -2 \quad x > 0$

$\therefore x > 0$

$3\log_2 2 = \log_2 (x+2)(x)$

$\log_2 2^3 = \log_2 (x^2 + 2x)$

$8 = x^2 + 2x$

$0 = x^2 + 2x - 8$

$0 = (x+4)(x-2)$

$x = -4$ or $x = 2$

since $x > 0$, $\boxed{x = 2}$

verify:

$RS = \log_2(2+2) + \log_2 2$

$= \log_2 4 + \log_2 2$

$= 2 + 1$

$= 3$

$= LS$

Since $x > -3$, $x = -6$ is an extraneous root.
Substitute $x = -1$ in the original equation to verify this solution.

$$\begin{aligned} \text{L.S.} &= \log_6(x + 3) + \log_6(x + 4) \qquad \text{R.S.} = 1 \\ &= \log_6(-1 + 3) + \log_6(-1 + 4) \\ &= \log_6 2 + \log_6 3 \\ &= \log_6(2)(3) \\ &= \log_6 6 \\ &= 1 \end{aligned}$$

Since the left side is equal to the right side, the solution is verified.

In Lesson 5.3, algebra was used to solve exponential equations for which both sides of an equation could be written with the same base. Most exponential equations cannot be written this way. Logarithms can be used to solve these equations.

Example 3 Using Logarithms to Solve Exponential Equations

Solve each exponential equation algebraically.
Give the solution to the nearest hundredth.

a) $9^x = 50$ **b)** $2(5^{x-2}) = 100$ **c)** $2^{x+3} = 6^{x-1}$

SOLUTION

a)

$9^x = 50$ Take the logarithm base 9 of each side.

$\log_9 9^x = \log_9 50$

$x = \log_9 50$ Apply the change of base formula.

$x = \dfrac{\log 50}{\log 9}$

$x \doteq 1.78$

b)

$2(5^{x-2}) = 100$ Divide each side by 2.

$5^{x-2} = 50$ Take the logarithm base 5 of each side.

$\log_5 5^{x-2} = \log_5 50$

$(x - 2) = \log_5 50$ Solve for x.

$x = \log_5 50 + 2$ Apply the change of base formula.

$x = \dfrac{\log 50}{\log 5} + 2$

$x \doteq 4.43$

Check Your Understanding

3. Solve each exponential equation algebraically. Give the solution to the nearest hundredth.

a) $12 = 4^x$ b) $36 = 3(2^{x+1})$
c) $3^{x+1} = 6^x$

a) $12 = 4^x$

$\log 12 = \log 4^x$

$\log 12 = x \log 4$

$\dfrac{\log 12}{\log 4} = x$

$x \doteq 1.79$

b) $36 = 3(2^{x+1})$

$12 = 2^{x+1}$

$\log 12 = \log 2^{x+1}$

$\log 12 = (x+1)\log 2$

$\dfrac{\log 12}{\log 2} = x + 1$

$x = \dfrac{\log 12}{\log 2} - 1$

$x \doteq 2.58$

c) $2^{x+3} = 6^{x-1}$ Take the common

c) $3^{x+1} = 6^x$

c) $2^{x+3} = 6^{x-1}$	Take the common logarithm of each side.
$\log 2^{x+3} = \log 6^{x-1}$	Apply the power law.
$(x + 3)\log 2 = (x - 1)\log 6$	Apply the distributive law.
$x\log 2 + 3\log 2 = x\log 6 - 1\log 6$	Collect like terms.
$3\log 2 + \log 6 = x\log 6 - x\log 2$	Apply the power law. Remove x as a common factor.
$\log 2^3 + \log 6 = x(\log 6 - \log 2)$	Apply the product and quotient laws.
$\log(8 \cdot 6) = x\log\left(\frac{6}{2}\right)$	
$\log 48 = x\log 3$	Solve for x.
$x = \frac{\log 48}{\log 3}$	
$x \doteq 3.52$	

c) $3^{x+1} = 6^x$

$\log 3^{x+1} = \log 6^x$

$(x+1)\log 3 = x\log 6$

$x\log 3 + \log 3 = x\log 6$

$\log 3 = x\log 6 - x\log 3$

$= x(\log 6 - \log 3)$

$= x\left(\log\left(\frac{6}{3}\right)\right)$

$\log 3 = x\log 2$

$\frac{\log 3}{\log 2} = x$

$x \doteq 1.58$

Assignment:
p.422 #5,7,(9-13)a

THINK FURTHER

In *Example 3c*, could logarithms with a different base be applied instead of common logarithms? Justify your answer. Why do you think common logarithms were used?

Discuss the Ideas

1. Why is it necessary to verify the solution of a logarithmic equation?

2. Is the root of a logarithmic equation always positive? Explain.