5.5 The Laws of Logarithms

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FOCUS Develop and use the laws of logarithms.

Get Started

Use the exponent laws to simplify each expression.

$2^5 \cdot 2^3$	$\frac{3^8}{3^2}$	$\frac{7^3}{7^6}$	$(7^5)^2$
$= 2^{5+3}$	$= 3^{8-2}$	$= 7^{3-6}$	$= 7^{5 \cdot 2}$
$= 2^8$	$= 3^6$	$= 7^{-3}$	$= 7^{10}$
		or $\frac{1}{7^3}$	

Construct Understanding

Use the exponent laws and the relationship between exponents and logarithms to complete each statement with a natural number. Describe your strategies. Use a calculator to check.

$\log 2 + \log 3 = \log ?$ $\quad \log 6 = \log 2 \cdot 3$

$\log 8 - \log 2 = \log ?$ $\quad \log 4 = \log 8 \div 2$

$3 \log 2 = \log ?$ $\quad \log 8 = \log 2^3$

For each statement above, write two more statements using the same operation.

Compare your results with those of your classmates.

Write rules for:

- adding two logarithms with the same base
- subtracting two logarithms with the same base
- multiplying a logarithm by an integer

Operations on logarithms with the same base obey the **laws of logarithms**.

Laws of Logarithms

When $x > 0$ and $y > 0$

Product law: $\log_b xy = \log_b x + \log_b y, b > 0, b \neq 1$

Quotient law: $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y, b > 0, b \neq 1$

Power law: $\log_b x^k = k \log_b x, b > 0, b \neq 1, k \in \mathbb{R}$

THINK FURTHER

In the power law for logarithms, why is $k \in \mathbb{R}$, while $b > 0$, $b \neq 1$?

The definition of a logarithm can be used to prove that the laws above are true for all logarithms.

Here is a proof of the product law.
To prove that $\log_b xy = \log_b x + \log_b y$:

Let $\log_b x = m$ and $\log_b y = n$	Apply the definition of a logarithm.
Then $x = b^m$ $\quad$ $y = b^n$	
So, $xy = b^m \cdot b^n$	Use the product rule for exponents.
$xy = b^{m+n}$	Write this exponential statement as a logarithmic statement.
$\log_b xy = m + n$	Substitute for m and n.
$\log_b xy = \log_b x + \log_b y$	

The proofs of the other two laws of logarithms are in the Exercises.

Example 1 Applying the Laws of Logarithms to Logarithms with Base 10

Use a law of logarithms to simplify each expression.
Use a calculator to verify the answer.

a) $\log 50 - \log 25$ **b)** $\log 5 + \log 12$ **c)** $3 \log 4$

SOLUTION

a) Use the quotient law.

$$\log 50 - \log 25 = \log\left(\frac{50}{25}\right)$$
$$= \log 2$$

Verify: $\log 50 - \log 25 = 0.3010\ldots$
$\log 2 = 0.3010\ldots$

b) Use the product law.

$$\log 5 + \log 12 = \log(5 \cdot 12)$$
$$= \log 60$$

Verify: $\log 5 + \log 12 = 1.7781\ldots$
$\log 60 = 1.7781\ldots$

c) Use the power law.

$$3 \log 4 = \log 4^3$$
$$= \log 64$$

Verify: $3 \log 4 = 1.8061\ldots$
$\log 4^3 = 1.8061\ldots$

Check Your Understanding

1. Simplify each expression. Use a calculator to verify the answer.

a) $\log 7 + \log 8$

b) $5 \log 2$

c) $\log 80 - \log 16$

a) $\log 7 \cdot 8 = \log 56$

b) $\log 2^5 = \log 32$

c) $\log\left(\frac{80}{16}\right) = \log 5$

Example 2 Using the Laws of Logarithms to Simplify Expressions

Write each expression as a single logarithm.

a) $2 \log x - \log y$

b) $\frac{1}{2} \log x - 3 \log y + 2 \log z$

c) $2 + \log_4 3$

SOLUTION

a) $2 \log x - \log y$ Use the power law to write $2 \log x$ as $\log x^2$.

$= \log x^2 - \log y$ Use the quotient law.

$= \log\left(\frac{x^2}{y}\right)$

Check Your Understanding

2. Write each expression as a single logarithm.

a) $\log x + 3 \log y$

b) $\log x + 2 \log y - 4 \log z$

c) $\log_2 6 - 3$

a) $\log x + 3 \log y$

$= \log x + \log y^3$

$= \log xy^3$

b) $\log x + 2\log y - 4\log z$

$= \log x + \log y^2 - \log z^4$

$= \log\left(\frac{xy^2}{z^4}\right)$

b) $\frac{1}{2}\log x - 3\log y + 2\log z$ Use the power law.

$= \log x^{\frac{1}{2}} - \log y^3 + \log z^2$ Use the quotient law.

$= \log\left(\frac{x^{\frac{1}{2}}}{y^3}\right) + \log z^2$ Use the product law.

$= \log\left(\frac{x^{\frac{1}{2}}z^2}{y^3}\right)$

c) $2 + \log_4 3$

Write 2 as a logarithm base 4:

$2 = \log_4 4^2$, or $\log_4 16$

So, $2 + \log_4 3 = \log_4 16 + \log_4 3$ Use the product law.

$= \log_4(16 \cdot 3)$

$= \log_4 48$

c) $\log_2 6 - 3$

$= \log_2 6 - 3\log_2 2$

$= \log_2 6 - \log_2 2^3$

$= \log_2 6 - \log_2 8$

$= \log_2\left(\frac{6}{8}\right)$

$= \log_2\left(\frac{3}{4}\right)$

Example 3 Writing a Logarithm as a Sum or Difference of Logarithms

Write each expression in terms of $\log a$, $\log b$, and/or $\log c$.

a) $\log a^2c$ **b)** $\log\left(\frac{a^2}{bc^3}\right)$

SOLUTION

a) $\log a^2c$ Use the product law.

$= \log a^2 + \log c$ Use the power law.

$= 2\log a + \log c$

b) $\log\left(\frac{a^2}{bc^3}\right)$ Use the quotient law.

$= \log a^2 - \log bc^3$ Use the power law and product law.

$= 2\log a - (\log b + \log c^3)$

$= 2\log a - \log b - \log c^3$ Use the power law.

$= 2\log a - \log b - 3\log c$

Check Your Understanding

3. Write each expression in terms of $\log a$, $\log b$, and/or $\log c$.

a) $\log\left(\frac{a}{b^2}\right)$ b) $\log\left(\frac{a^2\sqrt[3]{b}}{c}\right)$

a) $\log a - \log b^2$

$= \log a - 2\log b$

b) $\log a^2 + \log b^{1/3} - \log c$

$= 2\log a + \frac{1}{3}\log b - \log c$

Example 4 Using the Laws of Logarithms to Evaluate

Evaluate each expression.

a) $2 \log_2 6 - \log_2 9$ **b)** $\log_9 2 + 3 \log_9 6 - 4 \log_9 2$

SOLUTION

a)

$2 \log_2 6 - \log_2 9$	Use the power law.
$= \log_2 6^2 - \log_2 9$	
$= \log_2 36 - \log_2 9$	Use the quotient law.
$= \log_2\left(\frac{36}{9}\right)$	
$= \log_2 4$	Write the number as a power of 2.
$= \log_2 2^2$	Use $\log_b b^n = n$.
$= 2$	

b)

$\log_9 2 + 3 \log_9 6 - 4 \log_9 2$	Use the power law.
$= \log_9 2 + \log_9 6^3 - \log_9 2^4$	
$= \log_9 2 + \log_9 216 - \log_9 16$	Use the product law.
$= \log_9(2 \cdot 216) - \log_9 16$	Use the quotient law.
$= \log_9\left(\frac{2 \cdot 216}{16}\right)$	
$= \log_9 27$	Write the number as a power of 9.
$= \log_9(9 \cdot 3)$	
$= \log_9\left(9 \cdot 9^{\frac{1}{2}}\right)$	
$= \log_9 9^{\frac{3}{2}}$	
$= \frac{3}{2}$	

Check Your Understanding

4. Evaluate each expression.

a) $3 \log_9 6 - \log_9 72$

b) $2 \log_4 6 - 3 \log_4 3 + \log_4 12$

a) $3\log_9 6 - \log_9 72$

$= \log_9 6^3 - \log_9 72$

$= \log_9 216 - \log_9 72$

$= \log_9\left(\frac{216}{72}\right)$

$= \log_9 3$

$= \log_9 \sqrt{9}$

$= \log_9 9^{1/2}$

$= \frac{1}{2}\underbrace{\log_9 9}_{=1}$

$= \frac{1}{2}$

b) $2\log_4 6 - 3\log_4 3 + \log_4 12$

$= \log_4 6^2 - \log_4 3^3 + \log_4 12$

$= \log_4\left(\frac{6^2 \cdot 12}{3^3}\right)$

$= \log_4 16$

$= \log_4 4^2$

$= 2\log_4 4$

$= 2$

Assignment:

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