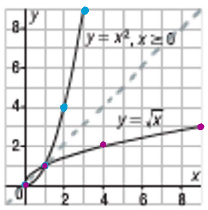
5.4 Logarithms and the Logarithmic Function

FOCUS Investigate logarithmic functions and relate them to exponential functions.

Get Started

How are these graphs related?



$y = x^2$	$y = \sqrt{x}$
(0,0)	(0,0)
(1,1)	(1,1)
(2,4)	(4,2)
(3,9)	(9,3)

- x and y values are switched
- reflection over the line $y = x$

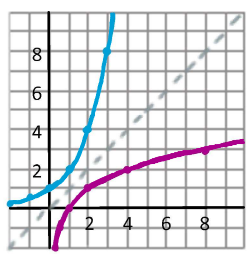
⇒ These graphs are inverses of each other.

Construct Understanding

Complete the table of values below for $y = 2^x$.
Use the completed table to graph $y = 2^x$ and its inverse.
What are the equations of the asymptotes for the graphs?
State the domain and the range of each function.

$y = 2^x$

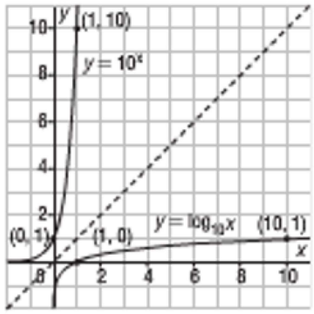
x	y
−2	1/4
−1	1/2
0	1
1	2
2	4
3	8



$(\frac{1}{4}, -2)$
$(\frac{1}{2}, -1)$
$(1, 0)$
$(2, 1)$
$(4, 2)$
$(8, 3)$

The term **logarithm** is used to describe the inverse of a power. For example, the inverse of 10^x is the logarithm to the base 10 of x, which is written as $\log_{10}x$. We say: log base 10 of x

Here is the graph of $y = 10^x$ and its inverse.



Each graph is a reflection of the other graph in the line $y = x$.
The inverse of $y = 10^x$ is $y = \log_{10}x$.

To understand what a logarithm is, consider the coordinates of corresponding points on the graphs of these functions:

$y = 10^x$

x	y	Points
0	$10^0 = 1$	(0, 1)
1	$10^1 = 10$	(1, 10)
2	$10^2 = 100$	(2, 100)

$y = \log_{10}x$

x	y	Points
1	$\log_{10}1 = 0$	(1, 0)
10	$\log_{10}10 = 1$	(10, 1)
100	$\log_{10}100 = 2$	(100, 2)

↑

$\log_{10}100 = 2$ means that 10 is raised to the power 2 to get 100; that is, $10^2 = 100$

Definition of a Logarithm

The logarithm of a number is an exponent.
$\log_b c$ is the power to which b is raised to get c.
The base of the logarithm is the same as the base of the power.

When $\log_b c = a$, then $c = b^a$, where $b > 0, b \neq 1, c > 0$

$\log_b c = a \quad \Leftrightarrow \quad b^a = c$

- the base doesn't change
- the values "a" and "c" switch between input and output

Example 1 Writing Expressions in Different Forms

a) Write each exponential expression as a logarithmic expression.

i) $2^5 = 32$ ii) $3^{-4} = \frac{1}{81}$ iii) $7^0 = 1$

b) Write each logarithmic expression as an exponential expression.

i) $\log_3 81 = 4$ ii) $\log_5 125 = 3$ iii) $\log_6 1 = 0$

SOLUTION

Use the definition of a logarithm.

a) i) $2^5 = 32$

The base is 2.
The logarithm is the exponent 5.
So, $5 = \log_2 32$

ii) $3^{-4} = \frac{1}{81}$

The base is 3.
The logarithm is the exponent -4.
So, $-4 = \log_3\left(\frac{1}{81}\right)$

iii) $7^0 = 1$

The base is 7.
The logarithm is the exponent 0.
So, $0 = \log_7 1$

b) i) $\log_3 81 = 4$

The base is 3.
The exponent is 4.
So, $81 = 3^4$

ii) $\log_5 125 = 3$

The base is 5.
The exponent is 3.
So, $125 = 5^3$

iii) $\log_6 1 = 0$

The base is 6.
The exponent is 0.
So, $6^0 = 1$

Check Your Understanding

1. a) Write each exponential expression as a logarithmic expression.

i) $3^3 = 27$

ii) $5^{-2} = \frac{1}{25}$

iii) $4^0 = 1$

b) Write each logarithmic expression as an exponential expression.

i) $\log_7 49 = 2$

ii) $\log_4\left(\frac{1}{64}\right) = -3$

iii) $\log_{10}\left(\frac{1}{10\,000}\right) = -4$

1.a) i) $3^3 = 27$ answer base input $\log_3 27 = 3$

ii) $\log_5\left(\frac{1}{25}\right) = -2$

iii) $\log_4 1 = 0$

b) i) $\log_7 49 = 2$ switch base $7^2 = 49$

ii) $4^{-3} = \frac{1}{64}$

iii) $10^{-4} = \frac{1}{10\,000}$

Since our number system is based on powers of 10, $\log_{10}x$ is called the **common logarithm** of x. When logarithms to base 10 are written, the base is often not shown; that is, $\log_{10}x$ is written as $\log x$.
On scientific and graphing calculators, use the LOG key to enter a logarithm with base 10.

For logarithms to bases other than 10, other strategies are used to evaluate them.
Consider these logarithms:
$\log_2 2^3$ is the power to which 2 is raised to get 2^3, which is 3.
So, $\log_2 2^3 = 3$
$\log_4 4^6$ is the power to which 4 is raised to get 4^6, which is 6.
So, $\log_4 4^6 = 6$
These examples illustrate this general result:

$$\log_b b^n = n$$

This equation and a related equation can be derived from the fact that $f(x) = \log_b x$ and $g(x) = b^x$ are inverses:
$f(g(n))$: $\log_b b^n = n$
$g(f(n))$: $b^{\log_b n} = n$
These equations can be used to simplify expressions involving exponents or logarithms.

$\log_b b^n = n$

Check Your Understanding

2. Evaluate each logarithm.

a) $\log_5 3125$

b) $\log_6\left(\frac{1}{216}\right)$

c) $\log_8(2\sqrt[3]{2})$

a) $3125 = 5^5$

$\log_5 3125 = \log_5 5^5 = 5$

b) $216 = 6^3 \therefore \frac{1}{216} = 6^{-3}$

$\log_6\left(\frac{1}{216}\right) = \log_6 6^{-3} = -3$

c) $\log_8(2\sqrt[3]{2})$

$2\sqrt[3]{2} = 2 \cdot 2^{1/3} = 2^{4/3}$

$2 = \sqrt[3]{8} = 8^{1/3}$

$2^{4/3} = (8^{1/3})^{4/3} = 8^{4/9}$

$\log_8(2\sqrt[3]{2}) = \log_8 8^{4/9}$

$= \frac{4}{9}$

Assignment:

p. 380 # 2, 5, 6a, 7, 9

Example 2 Evaluating Logarithms

Evaluate each logarithm.

a) $\log_3 729$ **b)** $\log_4\left(\frac{1}{32}\right)$ **c)** $\log_2(\sqrt[3]{4})$

SOLUTION

a) $\log_3 729$ Write 729 as a power of 3.

$\log_3 729 = \log_3 3^6$

$= 6$

b) $\log_4\left(\frac{1}{32}\right)$ Write $\frac{1}{32}$ as a power of 2.

$\log_4\left(\frac{1}{32}\right) = \log_4(2^{-5})$ Write 2^{-5} with a base of 4.

$= \log_4(2^2)^{-\frac{5}{2}}$

$= \log_4 4^{-\frac{5}{2}}$

$= -\frac{5}{2}$, or -2.5

c) $\log_2(\sqrt[3]{4})$ Write $\sqrt[3]{4}$ as a power of 2.

$\log_2(\sqrt[3]{4}) = \log_2 4^{\frac{1}{3}}$

$= \log_2 2^{\frac{2}{3}}$

$= \frac{2}{3}$

In *Example 2*, the logarithm of each number could be determined because the number could be written as a power of the base of the logarithm. If a number cannot be written this way, benchmarks can be used to estimate the value of a logarithm.

Example 3 — Using Benchmarks to Estimate the Value of a Logarithm

To the nearest tenth, estimate the value of $\log_2 10$.

SOLUTION

$\log_2 10$ has base 2, so use the powers of 2 closest to 10 as benchmarks:
$2^3 = 8$ and $2^4 = 16$
$\log_2 2^3 < \log_2 10 < \log_2 2^4$ $\qquad \log_b b^n = n$
So, $3 < \log_2 10 < 4$
10 is closer to 8, so $\log_2 10$ is likely closer to 3.
An estimate is: $\log_2 10 \doteq 3.3$
Check the estimate.
Calculate: $2^{3.3} = 9.8491\ldots$ This is less than 10, but close to 10.
Calculate: $2^{3.4} = 10.5560\ldots$ This is greater than 10, but not as close.
So, $\log_2 10 \doteq 3.3$

Check Your Understanding

3. To the nearest tenth, estimate the value of $\log_5 100$.

$5^2 = 25$; $5^3 = 125$

$\log_5 5^2 < \log_5 100 < \log_5 5^3$

$2 < \log_5 100 < 3$

estimate: 2.8

check: $5^{2.8} \doteq 90.597\ldots$

$5^{2.9} \doteq 106.417\ldots$

$\log_5 100 \doteq 2.9$

Example 4 — Identifying the Characteristics of the Graph of a Logarithmic Function

a) Graph $y = \log_3 x$.
b) Identify the intercepts, the equations of any asymptotes, and the domain and range of the function.

SOLUTION equivalent to $3^y = x$

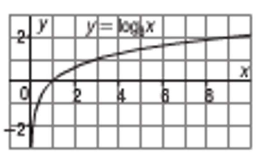
a) $y = \log_3 x$ is the inverse of $y = 3^x$, so construct a table of values for $y = 3^x$, then interchange the coordinates for a table of values for $y = \log_3 x$.

For $y = 3^x$

x	y
−2	$\frac{1}{9}$
−1	$\frac{1}{3}$
0	1
1	3
2	9

For $y = \log_3 x$

x	y
$\frac{1}{9}$	−2
$\frac{1}{3}$	−1
1	0
3	1
9	2



Check Your Understanding

4. a) Graph $y = \log_4 x$.

b) Identify the intercepts, the equations of any asymptotes, and the domain and range of the function.

$y = \log_4 x \Leftrightarrow 4^y = x$

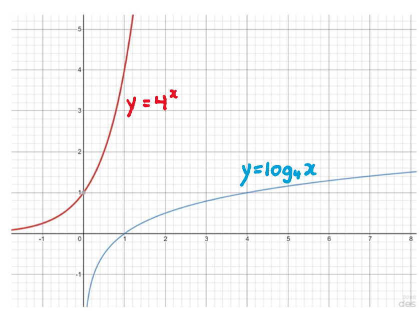
$\therefore$ inverse is $y = 4^x$

$y = 4^x$

x	y
−2	1/16
−1	1/4
0	1
1	4
2	16

$y = \log_4 x$

x	y
1/16	−2
1/4	−1
1	0
4	1
16	2



b) The graph does not intersect the y-axis, so it does not have a y-intercept.
The graph has x-intercept 1.
The y-axis is a vertical asymptote; its equation is $x = 0$.
The domain of the function is $x > 0$.
The range of the function is $y \in \mathbb{R}$.

The graph in *Example 4* illustrates the characteristics of a **logarithmic function**.

Definition of a Logarithmic Function

The logarithmic function $y = \log_b x$, $b > 0$, $b \neq 1$, is the inverse of the exponential function $y = b^x$.
The domain of $y = \log_b x$ is $x > 0$.

THINK FURTHER

In the definition of a logarithmic function, why is $b \neq 1$?

$\log_1 x = y \Leftrightarrow 1^y = x$
$\therefore x = 1$ for all values of y
This is a vertical line which is not a function.
$\therefore y = \log_1 x$ is not a function.

Assignment:
p.380 #2, 3, 11, 13

Discuss the Ideas

1. What is a logarithm? Explain what $\log_5 25$ means.

2. Why is it not possible to determine $\log_3(-27)$?

3. If $\log_b a < 0$ and $b > 1$, what can you say about a?