5.2 Analyzing Exponential Functions

FOCUS Sketch the graph of an exponential function and describe its characteristics.

Get Started

Suppose a colony of bacteria doubles in size every hour. Initially, there were 100 bacteria. How many bacteria would there be after each time?

- 3 h later

$100 \times 2 \times 2 \times 2 = 100 \times 2^3$

$= 800$ bacteria

- 5 h later

$100 \times 2^5 = 3200$ bacteria

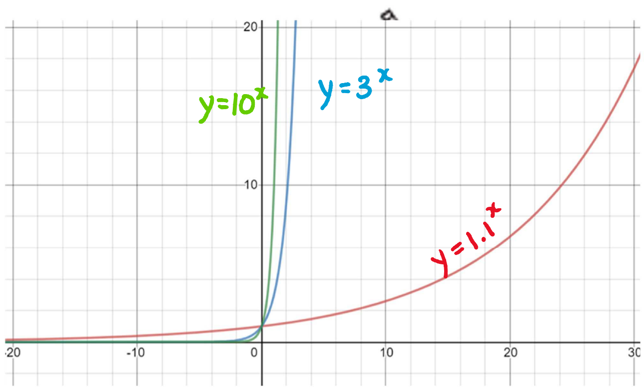
- 12 h later

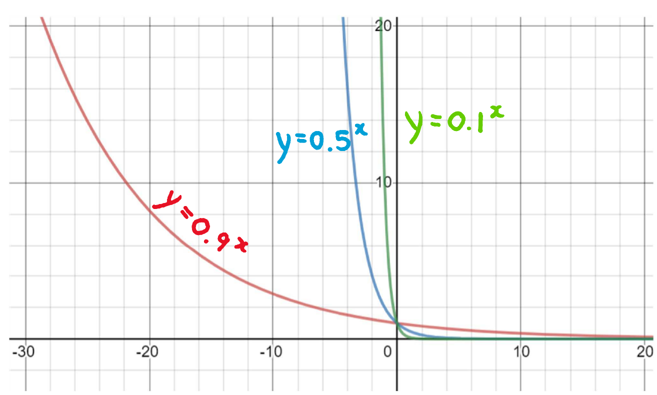
$100 \times 2^{12} = 409600$ bacteria

Construct Understanding

Use graphing technology. Graph each set of functions below on the same screen. For each set, how do the graphs change as the base of the power changes?
Which point do all 6 graphs have in common?
How are the graphs in set A different from the graphs in set B?

Set A:	$y = 1.1^x$	$y = 3^x$	$y = 10^x$
Set B:	$y = 0.9^x$	$y = 0.5^x$	$y = 0.1^x$

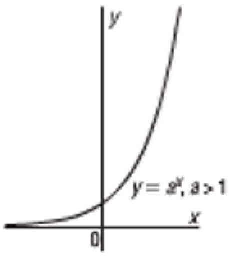




Exponential Function

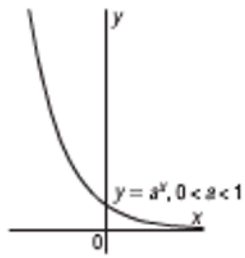
An exponential function is any function of x that can be written in the form $y = a^x$, where a is a positive constant.

Graph of $y = a^x$, $a > 1$



When $a > 1$, as x increases y increases.
The function is increasing.

Graph of $y = a^x$, $0 < a < 1$



When $0 < a < 1$, as x increases y decreases and approaches 0.
The function is decreasing.

An exponential function, $y = a^x$, $a > 0$, $a \neq 1$, has these characteristics:

- The graph has y-intercept 1.
- The graph approaches the x-axis, but never reaches it. The function has a horizontal asymptote with equation $y = 0$.
- The graph does not have an x-intercept.
- The domain of the function is $x \in \mathbb{R}$.
- The range of the function is $y > 0$.

THINK FURTHER

Is it possible for the graph of an exponential function $y = a^x$, $a > 0$, to have a y-intercept other than 1?

$x = 0$ at y-int

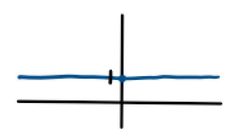
$a^0 = 1$

No, it is not possible.

THINK FURTHER

Describe the graph of the exponential function when $a = 1$.

$y = 1^x = 1$



Example 1 Sketching the Graph of an Exponential Function and Identifying Its Characteristics

a) Graph $y = 3^x$.

b) Determine:

i) the effect on y when x increases by 1
ii) whether the function is increasing or decreasing
iii) the intercepts
iv) the equations of any asymptotes
v) the domain of the function
vi) the range of the function

Check Your Understanding

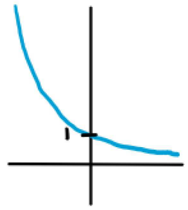
1. a) Graph $y = \left(\frac{1}{3}\right)^x$.

b) Determine:

i) the effect on y when x increases by 1
ii) whether the function is increasing or decreasing
iii) the intercepts
iv) the equations of any asymptotes
v) the domain of the function
vi) the range of the function

a)

x	$y=(\frac{1}{3})^x$
-2	9
-1	3
0	1
1	$1/3$
2	$1/9$



b) When x increases by 1, y is multiplied by $1/3$.

The function is decreasing

x-int: none

y-int: 1

asymptote: $y=0$ (x-axis)

D: $\{x \in \mathbb{R}\}$

R: $\{y \mid y>0, y \in \mathbb{R}\}$

p.349 #1-4,7

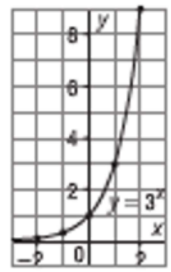
THINK FURTHER

How is the exponential function in *Example 1* related to a sequence?

SOLUTION

a) Create a table of values.

x	$y = 3^x$
-2	$\frac{1}{9}$
-1	$\frac{1}{3}$
0	1
1	3
2	9



b) i) From the table of values, when x increases by 1, y is multiplied by 3.

ii) Since y increases as x increases, the function is increasing.

iii) From the graph, there is no x-intercept. The y-intercept is 1.

iv) The x-axis is a horizontal asymptote; its equation is $y = 0$.

v) The domain of the function is $x \in \mathbb{R}$.

vi) The range of the function is $y > 0$.

The graph of an exponential function can be stretched, compressed, reflected in an axis, and translated, in that order.

The Function $y - k = ca^{d(x-h)}$, $a > 0$, $c \neq 0$, $d \neq 0$

When the graph of $y = a^x$ is:

- stretched vertically by a factor of $|c|$
- stretched horizontally by a factor of $\frac{1}{|d|}$
- reflected in the x-axis when $c < 0$
- reflected in the y-axis when $d < 0$
- translated k units vertically
- translated h units horizontally

the equation of the image graph is: $y - k = ca^{d(x-h)}$, $a > 0$

The general transformation is: (x, y) corresponds to $\left(\frac{x}{d} + h, cy + k\right)$

$y = ca^{d(x-h)} + k$

Example 2 Transforming the Graph of $y = a^x$

a) Use the graph of $y = 2^x$ to sketch the graph of $y = 2^{2x} - 4$.

b) For the function $y = 2^{2x} - 4$, determine:

i) whether the function is increasing or decreasing
ii) the intercepts
iii) the equation of the asymptote
iv) the domain of the function
v) the range of the function

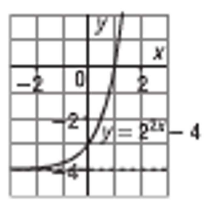
SOLUTION

a) Write the function as: $y + 4 = 2^{2x}$
Compare $y + 4 = 2^{2x}$ with $y - k = c2^{d(x-h)}$:
$k = -4, c = 1, d = 2$, and $h = 0$
The graph of $y = 2^{2x} - 4$ is the image of the graph of $y = 2^x$ after a horizontal compression by a factor of $\frac{1}{2}$, then a translation of 4 units down.

Use the general transformation: (x, y) corresponds to $\left(\frac{x}{d} + h, cy + k\right)$

The point (x, y) on $y = 2^x$ corresponds to the point $\left(\frac{x}{2}, y - 4\right)$ on $y + 4 = 2^{2x}$.
Choose points (x, y) on $y = 2^x$.

(x, y)	$\left(\frac{x}{2}, y - 4\right)$
$(-2, 0.25)$	$(-1, -3.75)$
$(-1, 0.5)$	$(-0.5, -3.5)$
$(0, 1)$	$(0, -3)$
$(1, 2)$	$(0.5, -2)$
$(2, 4)$	$(1, 0)$



b) From the graph of $y = 2^{2x} - 4$

i) The function is increasing.
ii) The graph has y-intercept -3 and x-intercept 1.
iii) The graph has a horizontal asymptote with equation $y = -4$.
iv) The domain of the function is $x \in \mathbb{R}$.
v) The range of the function is $y > -4$.

Check Your Understanding

2. a) Use the graph of $y = 2^x$ to sketch the graph of $y = 3(2^{-x+2})$.

b) From the graph of $y = 3(2^{-x+2})$, determine:

i) whether the function is increasing or decreasing
ii) the intercepts
iii) the equation of the asymptote
iv) the domain of the function
v) the range of the function

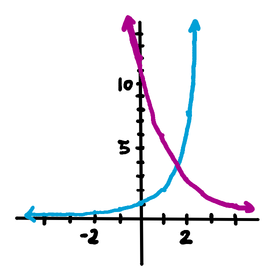
$y = 3(2^{-(x-2)})$

transformations:

- vertical expansion by a factor of 3 $y \to 3y$
- horizontal reflection (reflection in y-axis) $x \to -x$
- translation 2 units right $-x \to -x + 2$

$y = 2^x$

(x, y)	$(-x+2, 3y)$
$(-2, 1/4)$	$(4, 3/4)$
$(-1, 1/2)$	$(3, 3/2)$
$(0, 1)$	$(2, 3)$
$(1, 2)$	$(1, 6)$
$(2, 4)$	$(0, 12)$



The function is decreasing.
x-int: none
y-int: $y = 3(2^{0+2}) = 3(2^2) = 12$
asymptote: $y = 0$ *only changes with up/down translations
D: $\{x \in \mathbb{R}\}$
R: $\{y \mid y > 0, y \in \mathbb{R}\}$

Assignment: p. 349 #1-4, 7, 10

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