Example 2 **Determining a Composition of a Reciprocal Function and a Quadratic Function**

Given the functions $f(x) = \frac{1}{x-2}$ and $g(x) = x^2 - x$, determine an explicit equation for each composite function below, then state its domain.

a) $g(f(x))$ **b)** $f(g(x))$

SOLUTION

a) For $g(f(x))$:

In $g(x) = x^2 - x$, replace x with $\frac{1}{x-2}$.

$$g(f(x)) = \left(\frac{1}{x-2}\right)^2 - \left(\frac{1}{x-2}\right)$$

The domain of $f(x) = \frac{1}{x-2}$ is $x \neq 2$.

The domain of $g(x) = x^2 - x$ is $x \in \mathbb{R}$.

So, the domain of $g(f(x))$ is $x \neq 2$.

b) For $f(g(x))$:

In $f(x) = \frac{1}{x-2}$, replace x with $x^2 - x$.

$$f(g(x)) = \frac{1}{x^2 - x - 2}$$

The domain of $g(x) = x^2 - x$ is $x \in \mathbb{R}$.

The domain of $f(x) = \frac{1}{x-2}$ is $x \neq 2$.

So, $g(x)$ cannot equal 2; that is,

$x^2 - x \neq 2$

$x^2 - x - 2 \neq 0$

$(x + 1)(x - 2) \neq 0$

So, $x \neq -1$ and $x \neq 2$

So, the domain of $f(g(x))$ is $x \neq -1$ and $x \neq 2$.

Check Your Understanding

2. Given the functions $f(x) = \frac{1}{x+3}$ and $g(x) = x^2 - 4x$, determine an explicit equation for each composite function below, then state its domain.

a) $g(f(x))$ b) $f(g(x))$

a) $g(x) = x^2 - 4x$

$\frac{1}{x+3}$ $\frac{1}{x+3}$

$g(f(x)) = \left(\frac{1}{(x+3)}\right)^2 - \frac{4}{x+3}$

D: $x \neq -3$

b) $f(x) = \frac{1}{x+3}$

x^2-4x

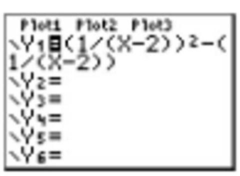
$f(g(x)) = \frac{1}{x^2-4x+3}$

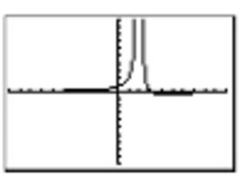
$x^2 - 4x + 3 = (x-1)(x-3)$

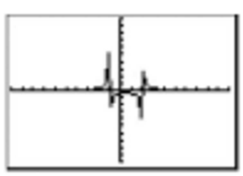
$= 0$ when $x = 1, 3$

D: $x \neq 1, 3$

The results of *Example 2* can be verified by using graphing technology.







Check Your Understanding

3. Given the functions $f(x) = \sqrt{x}$ and $g(x) = x^2 - 4$, determine an explicit equation for each composite function below, then state its domain.

a) $g(f(x))$ b) $f(g(x))$

a) $g(x) = x^2 - 4$

$\sqrt{x}$

$g(f(x)) = (\sqrt{x})^2 - 4$

$= x - 4$

D: $x \geq 0$

b) $f(x) = \sqrt{x}$

$x^2 - 4$

$f(g(x)) = \sqrt{x^2 - 4}$

$x^2 - 4 \geq 0$

$x^2 \geq 4$

$x \geq 2$ or $x \leq -2$

D: $x \leq -2$ or $x \geq 2$

Example 3 Determining a Composition of a Radical Function and a Quadratic Function

Given the functions $f(x) = \sqrt{x}$ and $g(x) = -x^2 + 2x$, determine an explicit equation for each composite function below, then state its domain.

a) $g(f(x))$ **b)** $f(g(x))$

SOLUTION

a) For $g(f(x))$:

In $g(x) = -x^2 + 2x$, replace x with $\sqrt{x}$.

$g(f(x)) = -(\sqrt{x})^2 + 2\sqrt{x}$

$g(f(x)) = -x + 2\sqrt{x}$

The domain of $f(x) = \sqrt{x}$ is $x \geq 0$.

The domain of $g(x) = -x^2 + 2x$ is $x \in \mathbb{R}$.

So, the domain of $g(f(x)) = -x + 2\sqrt{x}$ is $x \geq 0$.

b) For $f(g(x))$:

In $f(x) = \sqrt{x}$, replace x with $-x^2 + 2x$.

$f(g(x)) = \sqrt{-x^2 + 2x}$

The domain of $g(x) = -x^2 + 2x$ is $x \in \mathbb{R}$.

The domain of $f(x) = \sqrt{x}$ is $x \geq 0$.

So, $g(x)$ cannot be negative; that is,

$-x^2 + 2x \geq 0$

$x^2 - 2x \leq 0$

Solve the corresponding quadratic equation.

$x^2 - 2x = 0$

$x(x - 2) = 0$

$x = 0$ or $x = 2$

Choose a value of $x < 0$, such as $x = -1$.

Use mental math to substitute $x = -1$ in $x^2 - 2x \leq 0$.

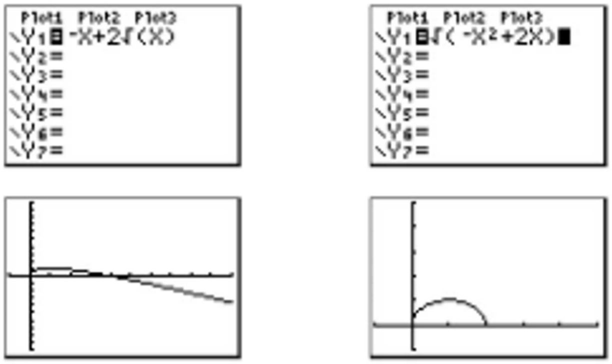
L.S. $= 3$ R.S. $= 0$

So, the interval $x < 0$ is not the correct interval.

That means that $0 \leq x \leq 2$ is the correct interval.

So, the domain of $f(g(x)) = \sqrt{-x^2 + 2x}$ is $0 \leq x \leq 2$.

The results of *Example 3* can be verified by using graphing technology.



Two functions that form a composite function may be determined by working backward.
Consider the quadratic function: $h(x) = 2(x - 3)^2 + 1$
To write $h(x)$ as a composite function $f(g(x))$, first identify a function of x contained within $h(x)$.
The binomial $(x - 3)$ is a function of x, and can be written as:
$g(x) = x - 3$
Then $h(x) = 2(g(x))^2 + 1$
Replace $g(x)$ with x to get: $f(x) = 2x^2 + 1$
So, $f(x) = 2x^2 + 1$ and $g(x) = x - 3$ form the composite function $f(g(x)) = 2(x - 3)^2 + 1$.

The function $f(g(x)) = 2(x - 3)^2 + 1$ can be considered as a composite function in a different way.
Consider $g(x) = (x - 3)^2$, then $f(g(x)) = 2g(x) + 1$
Replace $g(x)$ with x to get: $f(x) = 2x + 1$
So, $f(x) = 2x + 1$ and $g(x) = (x - 3)^2$ form the composite function $f(g(x)) = 2(x - 3)^2 + 1$.

Example 4 Writing a Function as a Composition of Two Functions

For each function, determine possible functions f and g so that $y = f(g(x))$.

a) $y = \frac{1}{x^2}$

b) $y = (3x^2 + 1)^4$

SOLUTION

a) Let $f(g(x)) = \frac{1}{x^2}$
Replace x^2 with x.
Then, $g(x) = x^2$ and $f(x) = \frac{1}{x}$

b) Let $f(g(x)) = (3x^2 + 1)^4$
Replace $3x^2 + 1$ with x.
Then, $g(x) = 3x^2 + 1$ and $f(x) = x^4$

Check Your Understanding

4. For each function, determine possible functions f and g so that $y = f(g(x))$.
a) $y = (x - 2)^3$
b) $y = \sqrt{3 + x}$

a) $f(x) = x^3$; $g(x) = x-2$

b) $f(x) = \sqrt{x}$; $g(x) = 3+x$

Assignment: p. 316 #6, 10