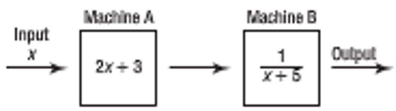
4.3 Composite Functions

4.3 Introduction to Composite Functions

FOCUS Determine the value of a composition of functions at a point and determine the equation of a composite function.

Get Started

Here are two function machines. The output of machine A is the input for machine B.



What is the output from machine B for each input in machine A?

- $x = 3$

$$\xrightarrow{3} 2(3)+3 = 9 \xrightarrow{9} \frac{1}{9+5} \to \frac{1}{14}$$

- $x = 0$

$$\xrightarrow{0} 2(0)+3 = 3 \xrightarrow{3} \frac{1}{3+5} \to \frac{1}{8}$$

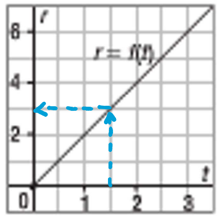
- $x = -2$

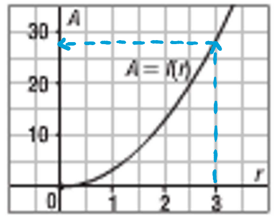
$$\xrightarrow{-2} 2(-2)+3 = -1 \xrightarrow{-1} \frac{1}{-1+5} \to \frac{1}{4}$$

Construct Understanding

When a drop of liquid falls on a flat surface, it forms a circle whose radius, r centimetres, is a function of time, t seconds, since the drop landed. The area of the circle, A square centimetres, is a function of

When a drop of liquid falls on a flat surface, it forms a circle whose radius, r centimetres, is a function of time, t seconds, since the drop landed. The area of the circle, A square centimetres, is a function of the radius, r centimetres.





How can the graphs above be used to determine the approximate area of the circle after 1.5 s?
Write an explicit equation for each function graphed above.
What is an explicit equation for A as a function of t?
Use the equation to determine the area of the circle after 1.5 s, to the nearest tenth of a square centimetre.

After 1.5 s the radius is 3 cm.
The area is about 28 cm².

$A(r) = \pi r^2$
$r(t) = 2t$

$A(r(t)) = \pi(2t)^2$
$= 4\pi t^2$

$A = 4\pi(1.5)^2 \doteq 28.3 \text{ cm}^2$

For a vehicle, the cost of gasoline, c dollars, is a function of the amount used, v litres.

For a vehicle, the cost of gasoline, c dollars, is a function of the amount used, v litres.
This can be written as the function: $c(v)$
The amount of fuel used, v, is a function of the distance driven, d kilometres.
This can be written as the function: $v(d)$
Since c is a function of v and v is a function of d, c can be written as a function of d: $c(v(d))$
The function $c(v(d))$ represents the cost of gasoline as a function of the distance driven.

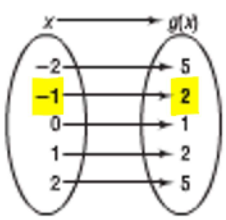
This is an example of a **composite function**; $c(v(d))$ is the composition of functions c and v; it is the function that results when function c is applied to function v.

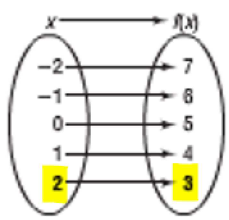
Composition of Functions

Given f and g are two functions of x, the composition of f and g is: $f(g(x))$, or $f \circ g(x)$
Both expressions are read as: "f of g at x."

The definition above illustrates that functions may simply be described by single letters, such as f and g.

Consider the two functions partially described by the arrow diagrams below.

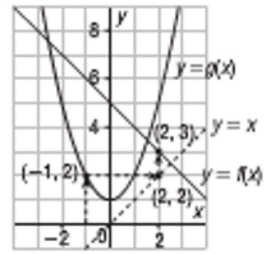




To determine $f(g(-1))$, start with the "inside" function g and determine $g(-1)$.
From the first arrow diagram, $g(-1) = 2$
So, $f(g(-1)) = f(2)$
From the second arrow diagram, $f(2) = 3$
So, $f(g(-1)) = 3$

The composition on page 293 can be illustrated graphically.
Here are the graphs of $y = f(x)$, $y = g(x)$, and $y = x$.

The composition on page 293 can be illustrated graphically. Here are the graphs of $y = f(x)$, $y = g(x)$, and $y = x$, for $x \in \mathbb{R}$. To determine $f(g(-1))$:



- Draw a vertical line through $x = -1$ to intersect the graph of $y = g(x)$; the point of intersection is $(-1, 2)$.
- From the point $(-1, 2)$, draw a horizontal line to intersect the graph of $y = x$; the point of intersection is $(2, 2)$. The y-value of $g(x)$ is the x-value of $f(x)$.
- From the point $(2, 2)$, draw a vertical line to intersect the graph of $y = f(x)$; the point of intersection is $(2, 3)$. So, $f(g(-1)) = 3$

This is called a *web graph*.

Check Your Understanding

1. Use the functions in *Example 1* to determine each value.

a) $g(f(2))$ b) $g(g(2))$

a) $g(f(2))$
$= g(0)$
$= 1$

b) $g(g(2))$
$= g(-1)$
$= 2$

extra practice:
p. 298 #4

Example 1 Determining the Value of a Composite Function Using Tables of Values

The tables below define two functions.
Use these tables to determine each value below.

x	$f(x)$
−2	8
−1	3
0	0
1	−1
2	0

x	$g(x)$
−2	3
−1	2
0	1
1	0
2	−1

a) $f(g(-1))$ **b)** $f(f(1))$

SOLUTION

a) $f(g(-1))$

First determine: $g(-1) = 2$

Then, $f(g(-1)) = f(2)$
$= 0$

b) $f(f(1))$

First determine: $f(1) = -1$

Then, $f(f(1)) = f(-1)$
$= 3$

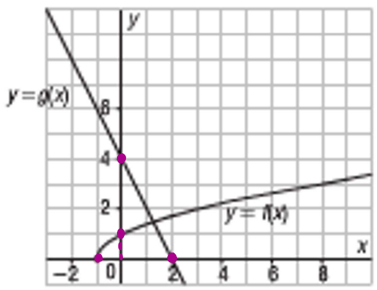
THINK FURTHER

In *Example 1*, why can the value of $f(f(-2))$ not be determined with any certainty?

Example 2 Determining the Value of a Composite Function Graphically

Example 2 Determining the Value of a Composite Function Graphically

Given the graphs of $y = f(x)$ and $y = g(x)$, determine each value below.



a) $f(g(-2))$ **b)** $g(f(3))$

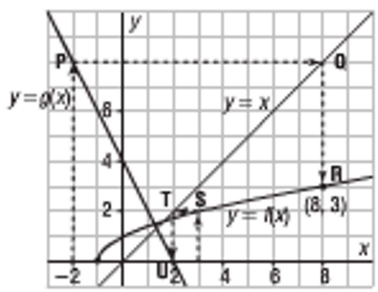
SOLUTION

On the graph, draw the line $y = x$.

a) $f(g(-2))$

Draw the line $x = -2$, which intersects the graph of $y = g(x)$ at P. From P, draw a horizontal line, which intersects the graph of $y = x$ at Q. From Q, draw a vertical line, which intersects the graph of $y = f(x)$ at R(8, 3).

So, $f(g(-2)) = 3$



b) $g(f(3))$

Draw the line $x = 3$, which intersects the graph of $y = f(x)$ at S. From S, draw a horizontal line, which intersects the graph of $y = x$ at T. From T, draw a vertical line, which intersects the graph of $y = g(x)$ at U(2, 0).

So, $g(f(3)) = 0$

Check Your Understanding

2. Use the graph in *Example 2* to determine each value.

a) $f(g(2))$ **b)** $g(f(-1))$

a) $f(g(2))$
$= f(0)$
$= 1$

b) $g(f(-1))$
$= g(0)$
$= 4$

extra practice:
p.299 #5

THINK FURTHER

In *Example 2a*, why is it possible to determine $f(g(-2))$ when the function f is not defined for $x = -2$?

Example 3 Determining the Value of a Composite Function Algebraically

Given the functions $f(x) = |x - 1|$ and $g(x) = \dfrac{1}{x^2 + 1}$, determine each value.

a) $f(g(2))$ **b)** $g(f(2))$

SOLUTION

a) $f(g(2))$

To determine $g(2)$, substitute $x = 2$ in

$$g(x) = \frac{1}{x^2 + 1}$$
$$g(2) = \frac{1}{2^2 + 1}$$
$$= 0.2$$

So, $f(g(2)) = f(0.2)$

Substitute $x = 0.2$ in

$$f(x) = |x - 1|$$
$$f(0.2) = |0.2 - 1|$$
$$= 0.8$$

So, $f(g(2)) = 0.8$

b) $g(f(2))$

To determine $f(2)$, substitute $x = 2$ in

$$f(x) = |x - 1|$$
$$f(2) = |2 - 1|$$
$$= 1$$

So, $g(f(2)) = g(1)$

Substitute $x = 1$ in

$$g(x) = \frac{1}{x^2 + 1}$$
$$g(1) = \frac{1}{1^2 + 1}$$
$$= 0.5$$

So, $g(f(2)) = 0.5$

Check Your Understanding

3. Given the functions $f(x) = x^2 + 3x$ and $g(x) = -2x + 1$, determine each value.

a) $f(g(9))$ b) $g(f(9))$

a) $g(9) = -2(9) + 1 = -17$

$f(g(9))$

$= f(-17)$

$= (-17)^2 + 3(-17)$

$= 289 - 51$

$= 238$

b) $f(9) = 9^2 + 3(9)$

$= 81 + 27$

$= 108$

$g(108) = -2(108) + 1$

$= -216 + 1$

$= -215$

extra practice:

p. 299 #6

In *Example 3*:

$f(g(2))$ is the value of $f(x)$ when $x = g(2)$.

$g(f(2))$ is the value of $g(x)$ when $x = f(2)$.

This example illustrates that, in general, $f(g(a)) \neq g(f(a))$, where $a \in \mathbb{R}$.

For two functions, such as $f(x) = 3x - 4$ and $g(x) = x^2 + 5x$, a composite function, $h(x)$, is formed by replacing x in $f(x)$ with $g(x)$.

That is, $h(x) = f(g(x))$

Begin with:

$f(x) = 3x - 4$	Replace x with $g(x)$.
$f(g(x)) = 3(g(x)) - 4$	On the right side, replace $g(x)$ with $x^2 + 5x$.
$f(g(x)) = 3(x^2 + 5x) - 4$	Simplify.
So, $h(x) = 3x^2 + 15x - 4$	

Consider the function: $f(x) = 2x + 3$, or $y = 2x + 3$

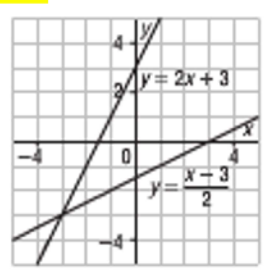
The inverse function is:

$x = 2y + 3$ Solve for y.

$2y = x - 3$

$y = \dfrac{x - 3}{2}$

inverses



Let the inverse function be $g(x)$.

Then, $g(x) = \frac{x - 3}{2}$

So, $f(g(x)) = 2\left(\frac{x - 3}{2}\right) + 3$
$= x$

And, $g(f(x)) = \frac{(2x + 3) - 3}{2}$
$= x$

In *Example 3*, it was shown that, in general, $f(g(x))$ and $g(f(x))$ are usually different. The example above illustrates that if two functions $f(x)$ and $g(x)$ are inverses of each other, then $f(g(x)) = g(f(x)) = x$ for all values of x in the domains of both functions.

Example 4 Writing the Equation of a Composite Function

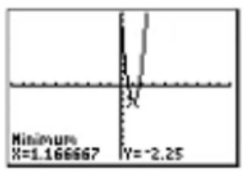
Given $f(x) = x^2 + 3x$ and $g(x) = 3x - 5$, determine an explicit equation for each composite function, then state its domain and range.

a) $f(g(x))$ **b)** $g(f(x))$ **c)** $f(f(x))$

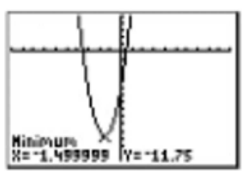
SOLUTION

Use $f(x) = x^2 + 3x$ and $g(x) = 3x - 5$.
Use graphing technology to determine the range.

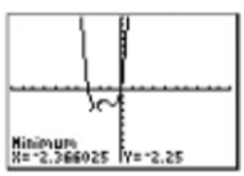
a) $f(g(x)) = f(3x - 5)$
In $f(x) = x^2 + 3x$, replace x with $3x - 5$.
$f(g(x)) = (3x - 5)^2 + 3(3x - 5)$
$f(g(x)) = 9x^2 - 30x + 25 + 9x - 15$
$f(g(x)) = 9x^2 - 21x + 10$
This is a quadratic function. Its domain is: $x \in \mathbb{R}$
From the graph, the range is: $y \geq -2.25$



b) $g(f(x)) = g(x^2 + 3x)$
In $g(x) = 3x - 5$, replace x with $x^2 + 3x$.
$g(f(x)) = 3(x^2 + 3x) - 5$
$g(f(x)) = 3x^2 + 9x - 5$
This is a quadratic function.
Its domain is: $x \in \mathbb{R}$
From the graph, the range is: $y \geq -11.75$



c) $f(f(x)) = f(x^2 + 3x)$
In $f(x) = x^2 + 3x$, replace x with $x^2 + 3x$.
$f(f(x)) = (x^2 + 3x)^2 + 3(x^2 + 3x)$
$f(f(x)) = x^4 + 6x^3 + 9x^2 + 3x^2 + 9x$
$f(f(x)) = x^4 + 6x^3 + 12x^2 + 9x$
This is a quartic function. Its domain is: $x \in \mathbb{R}$
From the graph, the minimum points have the same y-coordinate, and the range is: $y \geq -2.25$



Check Your Understanding

4. Given $f(x) = 2x^2 + 1$ and $g(x) = 2x + 7$, determine an explicit equation for each composite function, then state its domain and range.

a) $f(g(x))$ **b)** $g(f(x))$
c) $g(g(x))$

a) $f(2x+7) = 2(2x+7)^2 + 1$
$= 2(4x^2+28x+49) + 1$
$= 8x^2+56x+99$
D: $x \in \mathbb{R}$ R: $y \geq 1$

b) $g(2x^2+1) = 2(2x^2+1)+7$
$= 4x^2+9$
D: $x \in \mathbb{R}$ R: $y \geq 9$

c) $g(2x+7) = 2(2x+7)+7$
$= 4x+21$
D: $x \in \mathbb{R}$ R: $y \in \mathbb{R}$

Assignment:
p. 300 #7-10, 13, 14