

Rational Exponents

Today we will answer questions such as:

What does $2^{\frac{1}{2}}$ (2 to the power of $\frac{1}{2}$) mean? What about $4^{\frac{1}{2}}$? What about $8^{\frac{1}{3}}$ or $27^{\frac{2}{3}}$?

We can try answer these questions a few different ways.

1) Using patterns.

$$\begin{array}{ccc}
 \begin{array}{l} \sqrt{4} = 2 \\ \downarrow \\ 4^0 = 1 \\ \downarrow \times 2 \\ 4^{\frac{1}{2}} = 2 \\ \downarrow \times 2 \\ 4^1 = 4 \\ \downarrow \times 2 \\ 4^{\frac{3}{2}} = 8 \\ \downarrow \times 2 \\ 4^2 = 16 \end{array} &
 \begin{array}{l} \sqrt{9} = 3 \\ \downarrow \\ 9^0 = 1 \\ \downarrow \times 3 \\ 9^{\frac{1}{2}} = 3 \\ \downarrow \times 3 \\ 9^1 = 9 \\ \downarrow \times 3 \\ 9^{\frac{3}{2}} = 27 \\ \downarrow \times 3 \\ 9^2 = 81 \end{array} &
 \begin{array}{l} \sqrt{2} \\ \downarrow \\ 2^0 = 1 \\ \downarrow \times \sqrt{2} \\ 2^{\frac{1}{2}} = \sqrt{2} \\ \downarrow \times \sqrt{2} \\ 2^1 = 2 \\ \downarrow \times \sqrt{2} \\ 2^{\frac{3}{2}} = 2\sqrt{2} \\ \downarrow \times \sqrt{2} \\ 2^2 = 4 \end{array}
 \end{array}$$

2) Using exponent laws.

$$\begin{aligned}
 \frac{1}{2} + \frac{1}{2} &= 1 \\
 4^{\frac{1}{2}} \times 4^{\frac{1}{2}} &= 4^1 \\
 2 \times 2 &= 4 \\
 \Rightarrow 4^{\frac{1}{2}} &= 2
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{2} \times \frac{1}{2} &= \frac{1}{4} \\
 5^{\frac{1}{2}} \times 5^{\frac{1}{2}} &= 5^{\frac{1}{4}} \\
 \sqrt{5} \times \sqrt{5} &= 5 \\
 \Rightarrow 5^{\frac{1}{2}} &= \sqrt{5}
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} &= \frac{1}{27} \\
 8^{\frac{1}{3}} \times 8^{\frac{1}{3}} \times 8^{\frac{1}{3}} &= 8^{\frac{1}{27}} \\
 2 \times 2 \times 2 &= 8 \\
 \Rightarrow 8^{\frac{1}{3}} &= 2 \\
 8^{\frac{1}{3}} &= \sqrt[3]{8} = 2
 \end{aligned}$$

$$\begin{aligned}
 32^{\frac{1}{5}} \times 32^{\frac{1}{5}} \times 32^{\frac{1}{5}} \times 32^{\frac{1}{5}} \times 32^{\frac{1}{5}} &= 32^1 \\
 2 \times 2 \times 2 \times 2 \times 2 &= 32 \\
 \Rightarrow 32^{\frac{1}{5}} &= \sqrt[5]{32} = 2
 \end{aligned}$$

Examples: Evaluate the following:

$$81^{\frac{1}{2}} = \sqrt{81} = 9$$

$$16^{\frac{1}{4}} = \sqrt[4]{16} = 2$$

$$125^{\frac{1}{3}} = \sqrt[3]{125} = 5$$

$$\begin{aligned}
 16^{-\frac{1}{2}} &= \frac{1}{16^{\frac{1}{2}}} \\
 &= \frac{1}{\sqrt{16}} \\
 &= \frac{1}{4}
 \end{aligned}$$

How might we evaluate the following?

$$\begin{aligned}
 81^{\frac{3}{4}} &= (81^{1/4})^3 \quad \text{or} \quad (81^3)^{1/4} \\
 &= (\sqrt[4]{81})^3 \quad \xleftrightarrow{\text{same}} \quad \sqrt[4]{81^3} \\
 &= (3)^3 \\
 &= 27
 \end{aligned}$$

More Examples:

$$\begin{aligned}
 8^{\frac{2}{3}} & \\
 &= (8^{1/3})^2 \\
 &= (\sqrt[3]{8})^2 \\
 &= (2)^2 \\
 &= 4
 \end{aligned}$$

$$\begin{aligned}
 \left(\frac{9}{16}\right)^{\frac{3}{2}} & \\
 &= \left[\left(\frac{9}{16}\right)^{1/2}\right]^3 \\
 &= \left(\sqrt{\frac{9}{16}}\right)^3 \\
 &= \left(\frac{3}{4}\right)^3 \\
 &= \frac{27}{64}
 \end{aligned}$$

$$\begin{aligned}
 \left(\frac{9}{16}\right)^{\frac{-3}{2}} & \\
 &= \left(\frac{16}{9}\right)^{3/2} \\
 &= \left(\left(\frac{16}{9}\right)^{1/2}\right)^3 \\
 &= \left(\sqrt{\frac{16}{9}}\right)^3 \\
 &= \left(\frac{4}{3}\right)^3 \\
 &= \frac{64}{27}
 \end{aligned}$$

$$\begin{aligned}
 (-27)^{\frac{2}{3}} & \\
 &= \left((-27)^{1/3}\right)^2 \\
 &= (\sqrt[3]{-27})^2 \\
 &= (-3)^2 \\
 &= 9 \\
 \\
 (-8)^{\frac{2}{3}} & \\
 &= \left((-8)^{1/3}\right)^2 \\
 &= (\sqrt[3]{-8})^2 \\
 &= (-2)^2 \\
 &= 4
 \end{aligned}$$