3.5a Solving Exponential Equations Algebraically

5.3 Solving Exponential Equations

FOCUS Solve problems by modelling situations with exponential equations.

Get Started

$2(-3) = -6$

Solve the equation $2x^2 - 1x - 3 = 0$ algebraically.
How could you solve it graphically?

-6	sum
-1×6	5
-2×3	1
-3×2	-1 ✓

$$2x^2 - x - 3 = 0$$
$$2x^2 - 3x + 2x - 3 = 0$$
$$x(2x-3) + 1(2x-3) = 0$$
$$(2x-3)(x+1) = 0$$

$2x - 3 = 0$
$2x = 3$
$x = 3/2$

$x + 1 = 0$
$x = -1$

$$\boxed{x = \frac{3}{2}, -1}$$

Construct Understanding

Solve this equation in two ways: $2^x = 32$
Can you use the same two strategies to solve this equation? $2^x = 12$
Explain your response.
Solve the equation and write the root to the nearest hundredth.

Algebraic method: Write 32 as a power with base 2.

$32 = 2^5$

$$2^x = 32$$
$$2^x = 2^5$$
$$\therefore x = 5$$

THINK FURTHER

Is it possible for 2^x to be equal to any real number?

An **exponential equation** contains a power with a variable in the exponent.

An **exponential equation** contains a power with a variable in the exponent. For example, $3^x = 81$ is an exponential equation. One strategy to solve this equation uses the fact that when two powers with the same base are equal, their exponents are also equal.
Since 81 is a power of 3, both sides of the equation can be written as powers of 3.

$3^x = 81$
$3^x = 3^4$
So, $x = 4$

Example 1 Solving an Exponential Equation Using Common Bases

Solve each equation.

a) $2^x = \frac{1}{128}$ **b)** $4^x = 8^{x-1}$

SOLUTION

a) $2^x = \frac{1}{128}$ Write 128 as a power of 2.

$2^x = \frac{1}{2^7}$ Write the right side with a negative exponent.

$2^x = 2^{-7}$ Equate the exponents.

$x = -7$

b) $4^x = 8^{x-1}$

8 cannot be written as a power of 4; however, both 4 and 8 can be written as powers of 2.

$(2^2)^x = (2^3)^{x-1}$ Use exponent laws to simplify each side.

$2^{2x} = 2^{3x-3}$ Equate the exponents.

$2x = 3x - 3$

$x = 3$

The solutions of the equations in *Example 1* can be verified by substituting the root into each equation.
For example, in part b, for $4^x = 8^{x-1}$, substitute: $x = 3$

L.S.	R.S.
$= 4^x$	$= 8^{x-1}$
$= 4^3$	$= 8^{3-1}$
$= 64$	$= 8^2$
	$= 64$

Since the left side is equal to the right side, the solution is verified.

Check Your Understanding

1. Solve each equation.

a) $4^x = \frac{1}{256}$

b) $27^x = 9^{2x-1}$

a) $256 = 4^4$

$4^x = \frac{1}{256}$

$= \frac{1}{4^4}$

$4^x = 4^{-4}$

$\therefore x = -4$

b) $27^x = (3^3)^x = 3^{3x}$

$9^{2x-1} = (3^2)^{2x-1} = 3^{4x-2}$

$27^x = 9^{2x-1}$

$3^{3x} = 3^{4x-2}$

$\therefore 3x = 4x - 2$

$-4x \quad -4x$

$(-x = -2) \cdot (-1)$

$x = 2$

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Example 2 Solving an Exponential Equation Involving Radicals

Solve each equation.

a) $2^x = 4\sqrt{2}$ b) $(\sqrt{3})^{x-1} = \sqrt[3]{9}$

SOLUTION

a) $2^x = 4\sqrt{2}$ On the right side, write 4 as 2^2 and $\sqrt{2}$ as $2^{\frac{1}{2}}$.

$2^x = 2^2 \cdot 2^{\frac{1}{2}}$ Use the product of powers law.

$2^x = 2^{2+\frac{1}{2}}$

$2^x = 2^{\frac{5}{2}}$

$x = \frac{5}{2}$, or 2.5

b) $(\sqrt{3})^{x-1} = \sqrt[3]{9}$ Write both sides as powers of 3.

$\left(3^{\frac{1}{2}}\right)^{x-1} = \sqrt[3]{3^2}$

$\left(3^{\frac{1}{2}}\right)^{x-1} = 3^{\frac{2}{3}}$

$3^{\frac{1}{2}x-\frac{1}{2}} = 3^{\frac{2}{3}}$ Equate the exponents.

$\frac{1}{2}x - \frac{1}{2} = \frac{2}{3}$ Multiply by the common denominator 6.

$6\left(\frac{1}{2}x - \frac{1}{2}\right) = 6\left(\frac{2}{3}\right)$

$3x - 3 = 4$

$3x = 7$

$x = \frac{7}{3}$

Check Your Understanding

2. Solve each equation.

a) $2^x = 8\sqrt[3]{2}$

b) $(\sqrt{125})^{2x+1} = \sqrt[3]{625}$

a) $8\sqrt[3]{2} = 8 \cdot \sqrt[3]{2}$
$= 2^3 \cdot 2^{1/3}$
$= 2^{3+1/3}$
$= 2^{3\frac{1}{3}}$

$2^x = 8\sqrt[3]{2}$
$2^x = 2^{3\frac{1}{3}}$
$x = 3\frac{1}{3}$ or $\frac{10}{3}$

b) $125 = 5^3$; $625 = 5^4$

$(\sqrt{125})^{2x+1} = (125^{\frac{1}{2}})^{2x+1}$
$= ((5^3)^{\frac{1}{2}})^{2x+1}$
$= (5^{3/2})^{2x+1}$
$= 5^{3x+3/2}$

$\sqrt[3]{625} = 625^{1/3}$
$= (5^4)^{1/3}$
$= 5^{4/3}$

$(\sqrt{125})^{2x+1} = \sqrt[3]{625}$
$5^{3x+3/2} = 5^{4/3}$
$(3x + \frac{3}{2} = \frac{4}{3}) \cdot 6$
$18x + 9 = 8$
$-9 \quad -9$
$18x = -1$
$x = -\frac{1}{18}$

p. 365 #9ace, 10ace

When money is invested in a savings account, it earns interest. If the interest is reinvested in the account, it also earns interest; this is called **compound interest**.

Consider an investment of \$100 in a savings account that pays 2% annual interest, compounded annually. This means that the interest earned is calculated and paid into the account annually (once a year); that is, the *compounding period* is 1 year.

After 1 year, the amount is:

\$100 + 2% of \$100 = \$100 + 0.02(\$100)
= \$100(1 + 0.02), or \$100(1.02)

After 2 years, the amount is:

\$100(1.02) + 2% of \$100(1.02) = \$100(1.02)(1 + 0.02)
= $\$100(1.02)^2$

Each subsequent year, the amount increases by a factor of 1.02.

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$$\$100(1.02) + 2\% \text{ of } \$100(1.02) = \$100(1.02)(1 + 0.02)$$
$$= \$100(1.02)^2$$

Each subsequent year, the amount increases by a factor of 1.02.

After 3 years, the amount is: $\$100(1.02)^3$

This pattern continues.

After t years, the amount is: $\$100(1.02)^t$