3.5 Inverse Relations

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FOCUS Graph inverse relations and determine equations.

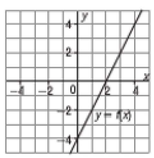
Get Started

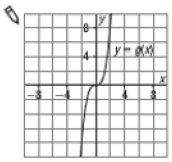
Solve each equation for x.

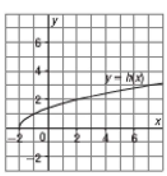
- $y = 3x - 4$
- $y = \frac{3x - 5}{2}$
- $y = 3x^2 - 5$
- $y = 2(x - 3)^2 + 4$

Construct Understanding

State the domain and range of each function below.
On each grid, reflect the graph in the line $y = x$.
Describe your strategy.
State the domain and range of the reflection image.
What is the relationship between the coordinates of pairs of corresponding points?
Use this relationship to describe a possible rule for reflecting a graph in the line $y = x$.







To determine the inverse equation of $y = 2x + 4$, switch x and y, then isolate y.

$y = 2x + 4$

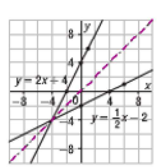
$\Rightarrow x = 2y + 4$

$-4 \qquad -4$

Here are the graphs of $y = 2x + 4$ and $y = \frac{1}{2}x - 2$.

The graph of $y = \frac{1}{2}x - 2$ is the image of the graph of $y = 2x + 4$ after a reflection in the line $y = x$.

Point on $y = 2x + 4$	Point on $y = \frac{1}{2}x - 2$



A point, such as $(-4, -4)$, which lies on the graphs

$y = 2x + 4$

$\Rightarrow x = 2y + 4$

$\quad -4 \qquad -4$

$\frac{x-4}{2} = \frac{2y}{2}$

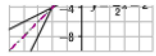
or $\frac{1}{2}(x-4) = y$

$\frac{1}{2}x - 2 = y$

line $y = x$.

Point on $y = 2x + 4$	Point on $y = \frac{1}{2}x - 2$
(1, 6)	(6, 1)
(0, 4)	(4, 0)
(−2, 0)	(0, −2)

The x and y values switch.



A point, such as (−4, −4), which lies on the graphs of $y = f(x)$ and $y = x$ is an *invariant point*. This point also lies on the graph of the inverse.

An invariant point does not move. (It doesn't vary.)

The coordinates of corresponding points are interchanged.

$y = \frac{1}{2}x - 2$ is the *inverse* of the function $y = 2x + 4$.

The equation of the inverse, $y = \frac{1}{2}x - 2$, can be solved for x and written as $x = 2y + 4$, or $x = f(y)$.

Reflecting in the Line $y = x$

For a function $y = f(x)$, the graph of $x = f(y)$ is the image of the graph of $y = f(x)$ after a reflection in the line $y = x$.
$y = f(x)$ and $x = f(y)$ are inverses of each other.
A point (x, y) on $y = f(x)$ corresponds to the point (y, x) on $x = f(y)$.

THINK FURTHER

How is the slope of the graph of a linear function related to the slope of its inverse?

When the inverse is also a function, the notation $f^{-1}(x)$ is used to denote the **inverse function**. We say, "*f inverse of x*."
For example, when $f(x) = 2x + 4$, then $f^{-1}(x) = \frac{1}{2}x - 2$

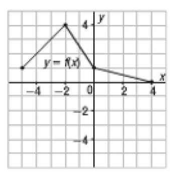
Example 1 Sketching the Inverse of a Function Given Its Graph

Here is the graph of $y = f(x)$.

a) Sketch the graph of its inverse on the same grid.

b) Is the inverse a function? Explain.

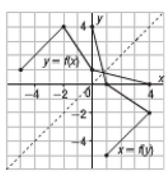
c) State the domain and range of the function and its inverse.



SOLUTION

a) Sketch the line $y = x$. Reflect points in this line.

Point on $y = f(x)$	Point on $x = f(y)$
(x, y)	(y, x)
(−5, 1)	(1, −5)
(−2, 4)	(4, −2)
(0, 1)	(1, 0)
(4, 0)	(0, 4)



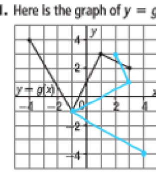
Plot the points, then join them with line segments.

b) Both points (1, 0) and (1, −5) lie on the graph of the inverse. So, the inverse is not a function because its graph does not pass the vertical line test.

c) The domain of $y = f(x)$ is: $-5 \le x \le 4$
The range of $y = f(x)$ is: $0 \le y \le 4$
The domain of $x = f(y)$ is: $0 \le x \le 4$
The range of $x = f(y)$ is: $-5 \le y \le 4$

Check Your Understanding

1. Here is the graph of $y = g(x)$.



(-4,4) → (4,-4)
(-1,-1) → (-1,-1)
(1,3) → (3,1)
(3,2) → (2,3)

a) Sketch the graph of its inverse on the same grid.

b) Is the inverse a function? Explain.

c) State the domain and range of the function and its inverse.

b) No, it is not a function because it fails the vertical line test.

c) D: $\{-4 \le x \le 3, x \in R\}$
R: $\{-1 \le y \le 4, y \in R\}$

D: $\{-1 \le x \le 4, x \in R\}$
R: $\{-4 \le y \le 3, y \in R\}$

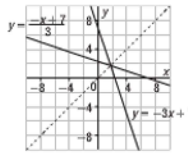
The domain and range have switched! (This is not a coincidence.)

The results from *Example 1* show a relationship between the domains and ranges of a function and its inverse.

Domain and Range of a Function and Its Inverse

The domain of $y = f(x)$ is the range of $x = f(y)$, and the range of $y = f(x)$ is the domain of $x = f(y)$.

To determine an equation of the inverse of a function, interchange x and y in the equation of the function, then solve the resulting equation for y.



To determine the equation of the inverse of $y = -3x + 7$:

$y = -3x + 7$ Interchange x and y.

$x = -3y + 7$ Solve for y.

$3y = -x + 7$

$y = \frac{-x + 7}{3}$

So, the equation of the inverse of $y = -3x + 7$ is $y = \frac{-x + 7}{3}$.

Example 2 Determining the Equation and Sketching the Graph of the Inverse of a Function

a) Determine an equation of the inverse of $y = x^2 + 3$.
b) Sketch graphs of $y = x^2 + 3$ and its inverse.
c) Is the inverse a function? Explain.

SOLUTION

a) To determine an equation of the inverse, interchange x and y then solve for y.

$y = x^2 + 3$
$x = y^2 + 3$
$y^2 = x - 3$
$y = \pm\sqrt{x - 3}$

An equation of the inverse is: $y = \pm\sqrt{x - 3}$

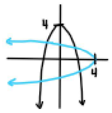
Check Your Understanding

2. a) Determine an equation of the inverse of $y = -x^2 + 4$.

b) Sketch graphs of $y = -x^2 + 4$ and its inverse.

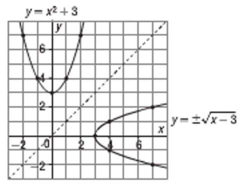
c) Is the inverse a function? Explain.

a) $x = -y^2 + 4$
$y^2 = -x + 4$
$y = \pm\sqrt{-x+4}$



The inverse is not a function.

b) The graph of $y = x^2 + 3$ is the image of the graph of $y = x^2$ after a translation of 3 units up. Sketch the graph of $y = x^2 + 3$, then reflect some points in the line $y = x$ by interchanging their coordinates. Join the points with a smooth curve.



c) Since the graph of the inverse does not pass the vertical line test, the inverse is not a function.

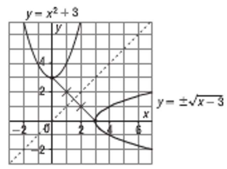
In *Example 2*, notice that the distance between a point on the graph of $y = f(x)$ and the line $y = x$ is the same as the distance between the corresponding point on the graph of $x = f(y)$ and the line $y = x$. This characteristic can be used to determine whether the graph of the inverse has been drawn correctly, or to determine whether two functions are inverses of each other.

If corresponding points are equidistant from the line $y = x$, then the midpoint of the line segment joining the points lies on the line $y = x$.

For example, in *Example 2*, to determine the midpoint of the line segment that joins corresponding points (0, 3) and (3, 0):

$$\text{Midpoint} = \left(\frac{0+3}{2}, \frac{3+0}{2}\right)$$
$$= \left(\frac{3}{2}, \frac{3}{2}\right)$$

Since this point lies on the line $y = x$, (0, 3) and (3, 0) are equidistant from the line. So, the graphs are likely inverses of each other.



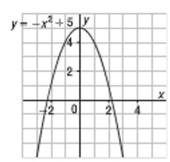
Examples 1 and *2* illustrate that the inverse of a function may not be a function. The domain of $f(x)$ can be restricted to ensure that its inverse is a function.

THINK FURTHER

When is the inverse of a function also a function?

Example 3 Determining Restrictions on the Domain of a Function

Determine two ways to restrict the domain of $y = -x^2 + 5$ so that its inverse is a function. Write the equation of the inverse each time.
Use a graph to illustrate each way.

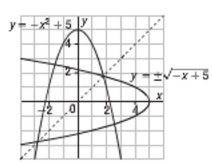


SOLUTION

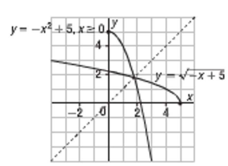
Sketch the graph of the inverse.
Determine an equation of the inverse:

$$x = -y^2 + 5$$
$$y^2 = -x + 5$$
$$y = \pm\sqrt{-x + 5}$$

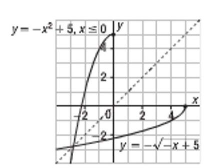
The graph of the inverse does not pass the vertical line test, so it is not a function.



The part of the graph on or above the x-axis does pass the vertical line test, so it does represent a function.
Its equation is $y = \sqrt{-x + 5}$ and its range is $y \geq 0$.
So, restrict the domain of $y = -x^2 + 5$ to $x \geq 0$.

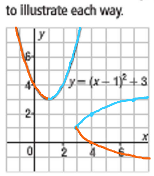


The part of the graph on or below the x-axis does pass the vertical line test, so it does represent a function.
Its equation is $y = -\sqrt{-x + 5}$ and its range is $y \leq 0$.
So, another way is to restrict the domain of $y = -x^2 + 5$ to $x \leq 0$.



Check Your Understanding

3. Determine two ways to restrict the domain of $y = (x - 1)^2 + 3$ so that its inverse is a function. Write the equation of the inverse each time. Use a graph to illustrate each way.



vertex @ (1,3)

① $x \geq 1$ or $x \leq 1$

$x = (y-1)^2 + 3$

$(y-1)^2 = x - 3$

$y - 1 = \pm\sqrt{x-3}$

$y = \pm\sqrt{x-3} + 1$

(this is before restrictions)

① The range of the inverse is $y \geq 1$.

$\therefore \quad y = \sqrt{x-3} + 1$

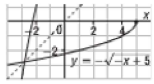
② The range of the inverse is $y \leq 1$.

∴ $y = \sqrt{x-3} + 1$

② The range of the inverse is $y \leq 1$.

∴ $y = -\sqrt{x-3} + 1$

its equation is $y = -\sqrt{-x} + 5$
and its range is $y \leq 0$.
So, another way is to restrict the domain of $y = -x^2 + 5$ to $x \leq 0$.



THINK FURTHER

In *Example 3*, are there other ways to restrict the domain so the inverse is a function? Explain.

Example 4 Determining Whether Two Functions Are Inverses of Each Other

a) Determine algebraically whether the functions in each pair are inverses of each other.

i) $y = 7x - 3$ and $y = \frac{x+3}{7}$

ii) $y = (x-2)^2 + 1, x \leq 2$ and $y = \sqrt{x-1} + 2$

b) Verify the answers to part a graphically.

SOLUTION

a) i) Determine an equation of the inverse of $y = 7x - 3$. Interchange x and y, then solve for y.

$$x = 7y - 3$$
$$x + 3 = 7y$$
$$y = \frac{x+3}{7}$$

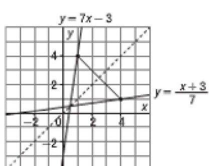
An equation of the inverse is: $y = \frac{x+3}{7}$
Since this matches the given equation, $y = 7x - 3$ and $y = \frac{x+3}{7}$ are inverses of each other.

ii) Determine an equation of the inverse of $y = (x-2)^2 + 1$, $x \leq 2$. Interchange x and y, then solve for y.

$$x = (y-2)^2 + 1$$
$$x - 1 = (y-2)^2$$
$$\pm\sqrt{x-1} = y - 2$$
$$y = \pm\sqrt{x-1} + 2$$

Since the domain of $y = (x-2)^2 + 1$ is $x \leq 2$, then the range of its inverse is $y \leq 2$.
So, an equation of the inverse is: $y = -\sqrt{x-1} + 2$
Since this equation is not equivalent to the given equation, $y = \sqrt{x-1} + 2$, the given functions are not inverses of each other.

b) i) Graph $y = 7x - 3$, $y = \frac{x+3}{7}$, and $y = x$ on the same grid.



The graphs appear to be reflections of each other in the line $y = x$.
To verify, determine the midpoint of the line segment that joins corresponding points (4, 1) and (1, 4):

Check Your Understanding

4. a) Determine algebraically whether the functions in each pair are inverses of each other.

i) $y = 3x - 6$ and $y = \frac{x - 6}{3}$ ← not inverses

$y = \frac{1}{3}(x-6)$ ← $y = \frac{x-6}{3}$

$y = \frac{1}{3}x - 2$

ii) $y = -x^2 + 3, x \geq 0$ and $y = \sqrt{3 - x}$ ← inverses

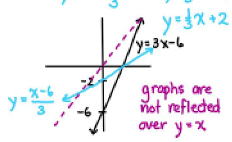
b) Verify the answers to part a graphically.

a) i) $x = 3y - 6$

$x + 6 = 3y$

$y = \frac{x+6}{3}$ or $y = \frac{1}{3}(x+6)$

$y = \frac{1}{3}x + 2$



graphs are not reflected over $y = x$

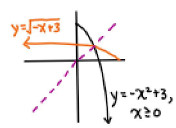
a) ii) $y = -x^2 + 3$, $x \geq 0$

$x = -y^2 + 3$

$y^2 = -x + 3$

$y = \sqrt{-x+3}$

(positive square root since range of inverse is $y \geq 0$)

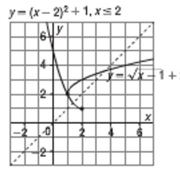


Assignment: #1, 2, 4, 5, 7, 8, 9a, 10a, 11

Midpoint $= \left(\frac{4+1}{2}, \frac{1+4}{2}\right)$, or $\left(\frac{5}{2}, \frac{5}{2}\right)$

Since this point lies on the line $y = x$, (4, 1) and (1, 4) are equidistant from the line. So, the graphs are likely inverses of each other.

ii) Graph $y = (x - 2)^2 + 1$, $x \leq 2$; $y = \sqrt{x - 1} + 2$; and $y = x$ on the same grid.



$y = (x - 2)^2 + 1, x \leq 2$, is part of the image of $y = x^2$ after a translation of 2 units right and 1 unit up. $y = \sqrt{x - 1} + 2$ is the image of $y = \sqrt{x}$ after a translation of 1 unit right and 2 units up.

Since the graphs are not reflections of each other in the line $y = x$, the functions are not inverses of each other.

Discuss the Ideas

1. Is the inverse of a linear function always a function? Is the inverse of a quadratic function with domain $x \in \mathbb{R}$ always a function? Explain.

2. What is the relationship between the domains and ranges of a function and its inverse? Explain.

3. What different strategies can you use to sketch the inverse of a function?