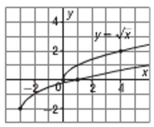
3.4 Combining Transformations of Functions

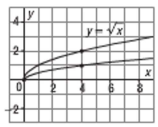
3.4 Combining Transformations of Functions

FOCUS **Apply translations and stretches to the graphs and equations of functions.**

Get Started

Two different transformation images of the graph of $y = \sqrt{x}$ are shown. Write an equation for each transformation image.



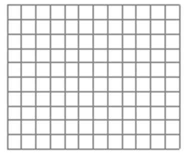


Construct Understanding

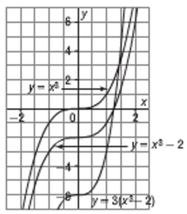
On the same grid, sketch the graph of each function below then describe it as a transformation image of the preceding graph.

- $y = |x|$
- $y = |2x|$
- $y = \frac{1}{4}|2x|$
- $y = \frac{1}{4}|2(x + 3)|$
- $y = \frac{1}{4}|2(x + 3)| + 4$

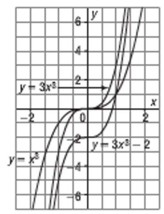
Describe how the graph of $y = \frac{1}{4}|2(x + 3)| + 4$ is related to the graph of $y = |x|$.



Here is the graph of $y = x^3$ and its images after a translation of 2 units down, followed by a vertical stretch by a factor of 3.



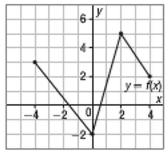
Here is the graph of $y = x^3$ and its images after a vertical stretch by a factor of 3, followed by a translation of 2 units down.



The same transformations are applied, but the image graphs are different. When two or more transformations are applied to a graph, the order in which the transformations are applied may make a difference; as shown above.

Example 1 — Applying a Combination of Transformations to the Graph of a Function

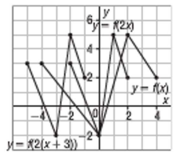
Here is the graph of $y = f(x)$. Sketch and label its image after a horizontal compression by a factor of $\frac{1}{2}$, then a translation of 3 units left.



SOLUTION

Perform the horizontal compression by a factor of $\frac{1}{2}$ first.

The point (x, y) on $y = f(x)$ corresponds to the point $\left(\frac{x}{2}, y\right)$ on the image graph $y = f(2x)$.

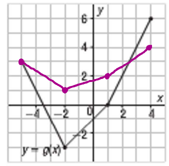


Point on $y = f(x)$	Point on $y = f(2x)$
$(-4, 3)$	$(-2, 3)$
$(0, -2)$	$(0, -2)$
$(2, 5)$	$(1, 5)$
$(4, 2)$	$(2, 2)$

Plot the points, then join them in order with line segments to form the graph of $y = f(2x)$. Then translate this graph 3 units left to form the graph of $y = f(2(x + 3))$.

Check Your Understanding

1. Here is the graph of $y = g(x)$.



Sketch and label its image after a vertical compression by a factor of $\frac{1}{3}$, then a translation of 2 units up.

① multiply y-values by $\frac{1}{3}$

② add 2 to y-values

$(-5,3) \rightarrow (-5, 1) \rightarrow (-5, 3)$

$(-2,-3) \rightarrow (-2,-1) \rightarrow (-2, 1)$

$(1,0) \rightarrow (1, 0) \rightarrow (1, 2)$

$(4,6) \rightarrow (4, 2) \rightarrow (4, 4)$

Does order matter? YES!

$3 \times \frac{1}{3} + 2 = 3$

$(3 + 2) \times \frac{1}{3} = \frac{5}{3}$

The results from Lessons 3.1 to 3.3 can be combined.

Combining Stretches, Compressions, and Translations

The graph of $y - k = af(b(x - h))$ is the image of the graph of $y = f(x)$ after these transformations:

- a horizontal stretch or compression by a factor of $\frac{1}{|b|}$;
- a reflection in the y-axis if $b < 0$;
- a vertical stretch or compression by a factor of $|a|$;
- a reflection in the x-axis if $a < 0$

Followed by:

- a horizontal translation of h units
- a vertical translation of k units

Point (x, y) on the graph of $y = f(x)$ corresponds to the point $\left(\frac{x}{b} + h, ay + k\right)$ on the graph of $y - k = af(b(x - h))$; this is the *general transformation*.

When graphing from an equation, to ensure the correct transformation image is sketched, apply the transformations in this order:

- stretches or compressions
- reflections
- translations

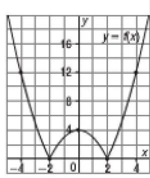
BEDMAS

THINK FURTHER

How do the expressions in $\left(\frac{x}{b} + h, ay + k\right)$ account for the order in which the transformations should be performed?

Example 2 Using the General Transformation to Sketch the Graph of a Function

Here is the graph of $y = f(x)$. Sketch the graph of $y - 6 = f(4(x + 2))$. State the domain and range of each function.



SOLUTION

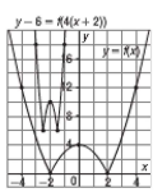
Compare: $y - k = af(b(x - h))$ to
$y - 6 = f(4(x + 2))$
$k = 6, a = 1, b = 4$, and $h = -2$

So, the graph of $y = f(x)$ is horizontally compressed by a factor of $\frac{1}{4}$, then translated 2 units left and 6 units up. Use the general transformation: (x, y) corresponds to $\left(\frac{x}{b} + h, ay + k\right)$

The point (x, y) on $y = f(x)$ corresponds to the point $\left(\frac{x}{4} - 2, y + 6\right)$ on $y - 6 = f(4(x + 2))$.

Choose some lattice points on $y = f(x)$, including the intercepts.

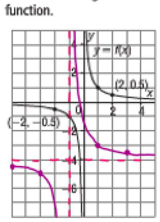
Point on $y = f(x)$	Point on $y - 6 = f(4(x + 2))$
(x, y)	$\left(\frac{x}{4} - 2, y + 6\right)$
(−4, 12)	(−3, 18)
(−2, 0)	(−2.5, 6)
(0, 4)	(−2, 10)
(2, 0)	(−1.5, 6)
(4, 12)	(−1, 18)



Plot the points, then draw a smooth curve through them.
The domain of $y = f(x)$ is: $x \in \mathbb{R}$
The range of $y = f(x)$ is: $y \geq 0$
The domain of $y - 6 = f(4(x + 2))$ is: $x \in \mathbb{R}$
The range of $y - 6 = f(4(x + 2))$ is: $y \geq 6$

Check Your Understanding

2. Here is the graph of $y = f(x)$. Sketch the graph of $y + 4 = f\left(\frac{1}{2}(x + 1)\right)$. State the domain and range of each function.



1. horizontal stretch by 2
→ multiply x-values by 2

2. no reflection

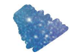
3. translation 4 units down and 1 unit left
→ subtract 4 from y-values and subtract 1 from x-values

(-2,-0.5) → (-4,-0.5) → (-5,-4.5)
(-1,-1) → (-2,-1) → (-3,-5)
(1,1) → (2, 1) → (1,-3)
(2,0.5) → (4,0.5) → (3,-3.5)

horizontal asymptote
y = 0 → y = -4
vertical asymptote

Plot the points, then draw a smooth curve through them.
The domain of $y = f(x)$ is: $x \in \mathbb{R}$
The range of $y = f(x)$ is: $y \geq 0$
The domain of $y - 6 = f(4(x + 2))$ is: $x \in \mathbb{R}$
The range of $y - 6 = f(4(x + 2))$ is: $y \geq 6$

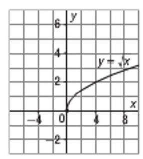
horizontal asymptote
$y = 0 \rightarrow y = -4$
vertical asymptote
$x = 0 \rightarrow x = -1$



In Chapter 2, you studied the graph of $y = \sqrt{x}$.
Example 3 illustrates 4 transformations of this function.

Example 3 Transforming the Graph of a Radical Function

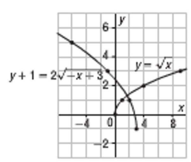
Use the graph of $y = \sqrt{x}$ to graph $y + 1 = 2\sqrt{-x + 3}$. What are the domain and range of $y + 1 = 2\sqrt{-x + 3}$?



SOLUTION

$y + 1 = 2\sqrt{-x + 3}$
To write the equation in the form $y - k = a\sqrt{b(x - h)}$, remove a common factor of -1 in the radicand.
$y + 1 = 2\sqrt{-1(x - 3)}$
Compare: $y - k = a\sqrt{b(x - h)}$ to $y + 1 = 2\sqrt{-1(x - 3)}$
$k = -1$, $a = 2$, $b = -1$, and $h = 3$
So, the graph of $y = \sqrt{x}$ is vertically stretched by a factor of 2, reflected in the y-axis, then translated 3 units right and 1 unit down.
Use the transformation: (x, y) on $y = \sqrt{x}$ corresponds to $(-x + 3, 2y - 1)$ on $y + 1 = 2\sqrt{-x + 3}$.
Choose points on $y = \sqrt{x}$.

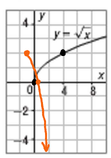
(x, y)	$(-x + 3, 2y - 1)$
(0, 0)	(3, −1)
(1, 1)	(2, 1)
(4, 2)	(−1, 3)
(9, 3)	(−6, 5)



Plot the points, then draw a smooth curve through them.
Domain of $y + 1 = 2\sqrt{-x + 3}$: $x \leq 3$
Range of $y + 1 = 2\sqrt{-x + 3}$: $y \geq -1$

Check Your Understanding

3. Use the graph of $y = \sqrt{x}$ to graph $y - 2 = -\sqrt{3x + 3}$. What are the domain and range of $y - 2 = -\sqrt{3x + 3}$?

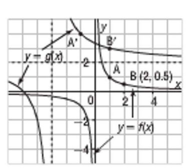


$y - 2 = -\sqrt{3(x+1)}$
① horizontal compression by $\frac{1}{3}$
→ multiply x-values by $\frac{1}{3}$
② vertical reflection (in x-axis)
→ y-values switch sign
③ translation up 2 and left 1
→ add 2 to y-values, subtract 1 from x-values

$(0,0) \rightarrow (0,0) \rightarrow (0,0) \rightarrow (-1,2)$
$(4,2) \rightarrow (\frac{4}{3},2) \rightarrow (\frac{4}{3},-2) \rightarrow (\frac{1}{3},0)$

Example 4 Writing the Equation of a Function after Transformations

The graph of $y = g(x)$ is the image of the graph of $y = f(x)$ after a combination of transformations. Corresponding points are labelled. Write then verify an equation for the image graph in terms of the function f.

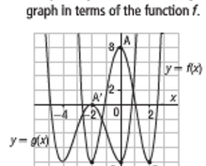


SOLUTION

Write the equation for the image graph in the form $y - k = af(b(x - h))$.
To identify the transformations, use A(1, 1) and B(2, 0.5).
The horizontal distance between A and B is: $|2 - 1| = 1$
The vertical distance between A and B is: $|0.5 - 1| = 0.5$
Use A′(−1, 4) and B′(1, 3):
The horizontal distance between A′ and B′ is: $|1 - (-1)| = 2$
The vertical distance between A′ and B′ is: $|3 - 4| = 1$
The horizontal distance doubles, so the graph of $y = f(x)$ is horizontally stretched by a factor of 2.
The vertical distance doubles, so the graph of $y = f(x)$ is vertically

Check Your Understanding

4. The graph of $y = g(x)$ is the image of the graph of $y = f(x)$ after a combination of transformations. Corresponding points are labelled. Write then verify an equation for the image graph in terms of the function f.



x-values decrease by 2
$(0,8) \rightarrow (-2,0)$

The horizontal distance between A′ and B′ is: $|1 - (-1)| = 2$
The vertical distance between A′ and B′ is: $|3 - 4| = 1$
The horizontal distance doubles, so the graph of $y = f(x)$ is horizontally stretched by a factor of 2.
The vertical distance doubles, so the graph of $y = f(x)$ is vertically stretched by a factor of 2.
So, $a = 2$ and $b = \frac{1}{2}$, or 0.5
To determine the coordinates of A(1, 1) after these stretches, substitute:
$x = 1, y = 1, a = 2$, and $b = 0.5$ in $\left(\frac{x}{b}, ay\right)$ to get $\left(\frac{1}{0.5}, 2\right)$, or (2, 2).

Determine the translation that would move (2, 2) to A′(−1, 4).
A translation of 3 units left and 2 units up is required, so $h = -3$ and $k = 2$.
An equation for the image graph in terms of $f(x)$ is:
$y - 2 = 2f(0.5(x + 3))$
Check:
Each point (x, y) on $y = f(x)$ corresponds to the point $(2x - 3, 2y + 2)$ on $y - 2 = 2f(0.5(x + 3))$.
So, the image of B(2, 0.5) is: $(2(2) - 3, 2(0.5) + 2)$, or B′(1, 3)
Since B′ lies on $y = g(x)$, the equation is likely correct.

x-values decrease by 2

$(0,8) \rightarrow (-2,0)$

$(2 -8) \rightarrow (0,-8)$

vertical compression by $\frac{1}{2}$
translation down by 4

$y = f(x+2)$

$y = \frac{1}{2}f(x+2) - 4$

Assignment:
#3-6, 9, 10