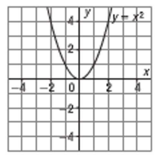
3.3 Stretching and Compressing Graphs

3.3 Stretching and Compressing Graphs of Functions

FOCUS **Relate changes in the equation of a function to stretches and compressions of its graph.**

Get Started

Here is the graph of $y = x^2$.
How is the graph of $y = ax^2$ related to the graph of $y = x^2$ for each value of a?



- $a = 0.5$

- $a = 2$

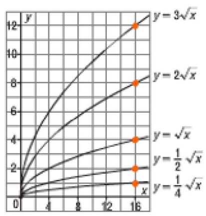
- $a = -0.5$

- $a = -2$

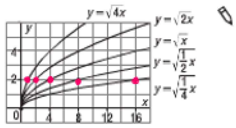
Construct Understanding

Look at the following graphs.
Each graph is the image of the graph of $y = \sqrt{x}$ after a transformation.
A point on the graph of $y = \sqrt{x}$ and its images are shown.
How does the graph of $y = a\sqrt{x}$ compare to the graph of $y = \sqrt{x}$?
What is the effect of a?
How does the graph of $y = \sqrt{bx}$ compare to the graph of $y = \sqrt{x}$?
What is the effect of b?
How does the graph of $y = a\sqrt{bx}$ compare to the graph of $y = \sqrt{x}$?
What are the effects of a and b?

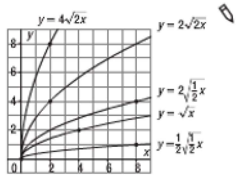
Graphs of the form $y = a\sqrt{x}$

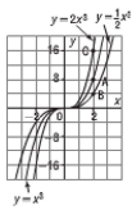


Graphs of the form $y = \sqrt{bx}$



Graphs of the form $y = a\sqrt{bx}$





Consider the graph of $y = x^3$.

If y is replaced with $2y$, the equation becomes $2y = x^3$, or $y = \frac{1}{2}x^3$.

If y is replaced with $\frac{1}{2}y$, the equation becomes $\frac{1}{2}y = x^3$, or $y = 2x^3$.

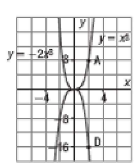
Look at the points where the graphs intersect the vertical line $x = 2$: A(2, 8), B(2, 4), C(2, 16)

The y-coordinate of point B is $\frac{1}{2}$ the y-coordinate of point A. This will be true for any point on $y = x^3$ and the point with the same x-coordinate on $y = \frac{1}{2}x^3$.

So, the graph of $y = \frac{1}{2}x^3$ is the image of the graph of $y = x^3$ after a vertical compression by a factor of $\frac{1}{2}$.

The y-coordinate of point C is 2 times the y-coordinate of point A. This will be true for any point on $y = x^3$ and the point with the same x-coordinate on $y = 2x^3$.

So, the graph of $y = 2x^3$ is the image of the graph of $y = x^3$ after a vertical stretch by a factor of 2.



Consider the graph of $y = x^3$.

If y is replaced with $-\frac{1}{2}y$, the equation becomes $-\frac{1}{2}y = x^3$, or $y = -2x^3$.

The y-coordinate of point D(2, −16) is −2 times the y-coordinate of point A(2, 8). This will be true for any point on $y = x^3$ and the point with the same x-coordinate on $y = -2x^3$.

So, the graph of $y = -2x^3$ is the image of the graph of $y = x^3$ after a vertical stretch by a factor of 2 and a reflection in the x-axis.

Vertical Stretches, Compressions, and Reflections

The graph of $y = af(x)$ is the image of the graph of $y = f(x)$ after a vertical stretch, compression, or reflection. Point (x, y) on $y = f(x)$ corresponds to point (x, ay) on $y = af(x)$.

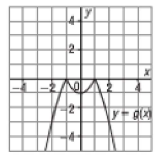
- When $0 < |a| < 1$, there is a vertical compression by a factor of $|a|$.

The graph of $y = af(x)$ is the image of the graph of $y = f(x)$ after a vertical stretch, compression, or reflection. Point (x, y) on $y = f(x)$ corresponds to point (x, ay) on $y = af(x)$.

- When $0 < |a| < 1$, there is a vertical compression by a factor of $|a|$.
- When $|a| > 1$, there is a vertical stretch by a factor of $|a|$.
- When $a < 0$, there is a reflection in the x-axis as well as the stretch or compression.

Example 1 Sketching the Graph of a Function after a Vertical Stretch and Reflection

Here is the graph of $y = g(x)$.
Sketch the graph of $y = -3g(x)$.
State the domain and range of each function.

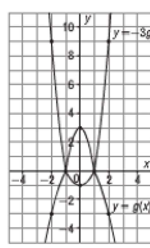


SOLUTION

Compare $y = ag(x)$ to $y = -3g(x)$:
$a = -3$
So, the graph of $y = g(x)$ is vertically stretched by a factor of 3, then reflected in the x-axis.
Use: (x, y) on $y = g(x)$ corresponds to $(x, -3y)$ on $y = -3g(x)$.
Choose the intercepts and some lattice points on $y = g(x)$.

multiply y-values by 3

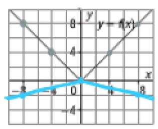
Point on $y = g(x)$	Point on $y = -3g(x)$
$(-2, -3)$	$(-2, 9)$
$(-1, 0)$	$(-1, 0)$
$(0, -1)$	$(0, 3)$
$(1, 0)$	$(1, 0)$
$(2, -3)$	$(2, 9)$



Plot the points, then draw a smooth curve through them.
Both functions have domain: $x \in \mathbb{R}$
The range of $y = g(x)$ is: $y \leq 0$
The range of $y = -3g(x)$ is: $y \geq 0$

Check Your Understanding

1. Here is the graph of $y = f(x)$.
Sketch the graph of
$y = -\frac{1}{4}f(x)$.
State the domain and range of each function.



multiply y-values by $-\frac{1}{4}$

$(-8,8) \rightarrow (-8,-2)$
$(-4,4) \rightarrow (-4,-1)$
$(0,0) \rightarrow (0,0)$
$(4,4) \rightarrow (4,-1)$
$(8,8) \rightarrow (8,-2)$

$y = f(x)$
$D: \{x \in \mathbb{R}\}$
$R: \{y \geq 0, y \in \mathbb{R}\}$

multiply by $-\frac{1}{4}$ (sign flips)

$y = -\frac{1}{4}f(x)$
$D: \{x \in \mathbb{R}\}$
$R: \{y \leq 0, y \in \mathbb{R}\}$

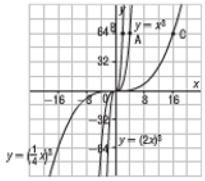
THINK FURTHER

In *Example 1*, the graph of $y = g(x)$ has equation $y = -|-x^2 + 1|$. What is the explicit equation for $y = -3g(x)$?

Consider the graph of $y = x^3$.
If x is replaced with $2x$, the equation becomes $y = (2x)^3$.

If x is replaced with $\frac{1}{4}x$, the equation becomes $y = \left(\frac{1}{4}x\right)^3$.

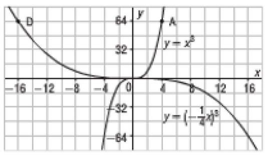
Look at the points where the graphs intersect the horizontal line $y = 64$:
A(4, 64), B(2, 64), C(16, 64)



The x-coordinate of point B is $\frac{1}{2}$ the x-coordinate of point A.
This will be true for any point on $y = x^3$ and the point with the same y-coordinate on $y = (2x)^3$. So, the graph of $y = (2x)^3$ is the image of the graph of $y = x^3$ after a horizontal compression by a factor of $\frac{1}{2}$.
The x-coordinate of point C is 4 times the x-coordinate of point A.
This will be true for any point on $y = x^3$ and the point with the same y-coordinate on $y = \left(\frac{1}{4}x\right)^3$. So, the graph of $y = \left(\frac{1}{4}x\right)^3$ is the image of the graph of $y = x^3$ after a horizontal stretch by a factor of 4.

Consider the graph of $y = x^3$.
If x is replaced with $-\frac{1}{4}x$, the equation becomes $y = \left(-\frac{1}{4}x\right)^3$.

The x-coordinate of point D$(-16, 64)$ is -4 times the x-coordinate of point A(4, 64).



This will be true for any point on $y = x^3$ and the point with the same y-coordinate on $y = \left(-\frac{1}{4}x\right)^3$. So, the graph of $y = \left(-\frac{1}{4}x\right)^3$ is the image of the graph of $y = x^3$ after a horizontal stretch by a factor of 4 and a reflection in the y-axis.

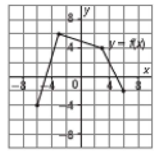
Horizontal Stretches, Compressions, and Reflections

The graph of $y = f(bx)$ is the image of the graph of $y = f(x)$ after a horizontal stretch, compression, or reflection. Point (x, y) on $y = f(x)$ corresponds to point $\left(\frac{x}{b}, y\right)$ on $y = f(bx)$.

- When $0 < |b| < 1$, there is a horizontal stretch by a factor of $\frac{1}{|b|}$.
- When $|b| > 1$, there is a horizontal compression by a factor of $\frac{1}{|b|}$.
- When $b < 0$, there is a reflection in the y-axis as well as the stretch or compression.

Example 2 Sketching the Graph of a Function after a Horizontal Stretch and Reflection

Here is the graph of $y = f(x)$. Sketch the graph of $y = f(-3x)$. State the domain and range of each function.



SOLUTION

Compare $y = f(bx)$ to $y = f(-3x)$: $b = -3$

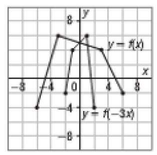
So, the graph of $y = f(x)$ is horizontally compressed by a factor of $\frac{1}{3}$, then reflected in the y-axis.

Use: (x, y) on $y = f(x)$ corresponds to $\left(\frac{x}{-3}, y\right)$ on $y = f(-3x)$.

The graph consists of line segments. Choose the endpoints of the line segments.

divide x-values by -3

Point on $y = f(x)$	Point on $y = f(-3x)$
$(-6, -4)$	$(2, -4)$
$(-3, 6)$	$(1, 6)$
$(3, 4)$	$(-1, 4)$
$(6, -2)$	$(-2, -2)$



Plot the points, then join them in order with line segments.

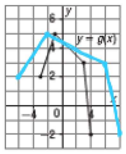
The domain of $y = f(x)$ is: $-6 \leq x \leq 6$

The domain of $y = f(-3x)$ is: $-2 \leq x \leq 2$

Both functions have range: $-4 \leq y \leq 6$

Check Your Understanding

2. Here is the graph of $y = g(x)$. Sketch the graph of $y = g(0.5x)$. State the domain and range of each function.



divide x-values by 0.5

$(-3,2) \rightarrow (-6,2)$

$(-1,5) \rightarrow (-2,5)$

$(0,4.5) \rightarrow (0,4.5)$

$(3,3) \rightarrow (6,3)$

$(4,-2) \rightarrow (8,-2)$

$y = g(x)$

D: $\{-3 \leq x \leq 4, x \in R\}$

R: $\{-2 \leq y \leq 5, y \in R\}$

divide by 0.5

$y = g(0.5x)$

D: $\{-6 \leq x \leq 8, x \in R\}$

R: $\{-2 \leq y \leq 5, y \in R\}$

Combining Transformations

Stretches, compressions, and reflections may be combined.

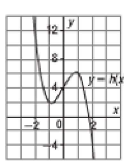
The point (x, y) on $y = f(x)$ corresponds to the point $\left(\frac{x}{b}, ay\right)$ on $y = af(bx)$.

Example 3 Sketching the Graph of a Function after a Combination of Transformations

Here is the graph of $y = h(x)$.

Sketch the graph of $y = -\frac{1}{2}h(2x)$.

State the domain and range of each function.



SOLUTION

Compare $y = ah(bx)$ to $y = -\frac{1}{2}h(2x)$:

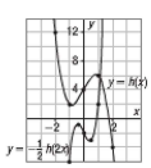
$a = -\frac{1}{2}$ and $b = 2$

So, the graph of $y = h(x)$ is vertically compressed by a factor of $\frac{1}{2}$, horizontally compressed by a factor of $\frac{1}{2}$, then reflected in the x-axis.

Use: (x, y) on $y = h(x)$ corresponds to $\left(\frac{x}{2}, -\frac{1}{2}y\right)$ on $y = -\frac{1}{2}h(2x)$.

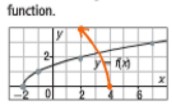
Choose lattice points on $y = h(x)$.

Point on $y = h(x)$	Point on $y = -0.5h(2x)$
$(-2, 12)$	$(-1, -6)$
$(-1, 2)$	$(-0.5, -1)$
$(0, 4)$	$(0, -2)$
$(1, 6)$	$(0.5, -3)$
$(2, -4)$	$(1, 2)$



Check Your Understanding

3. Here is the graph of $y = f(x)$. Sketch the graph of $y = 4f(-0.5x)$. State the domain and range of each function.



multiply y-values by 4

divide x-values by -0.5

$(-2,0) \rightarrow (4, 0)$

$(-1,1) \rightarrow (2, 4)$

$(2,2) \rightarrow (-4, 8)$

$(7,3) \rightarrow (-14, 12)$

$y = f(x)$

D: $\{x \geq -2, x \in R\}$

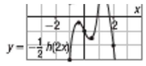
R: $\{y \geq 0, y \in R\}$

divide by -0.5 (sign flips)

multiply by 4

$y = 4f(-0.5x)$

(0, 4)	(0, −2)
(1, 6)	(0.5, −3)
(2, −4)	(1, 2)

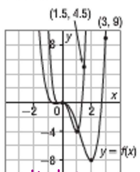


Plot the points, then join them with a smooth curve.
Both functions have domain: $x \in \mathbb{R}$
Both functions have range: $y \in \mathbb{R}$

D : $\{x \geq -2, x \in \mathbb{R}\}$ (sign flips)
R: $\{y \geq 0, y \in \mathbb{R}\}$
$y = 4f(-0.5x)$ multiply by 4
D: $\{x \leq 4, x \in \mathbb{R}\}$
R: $\{y \geq 0, y \in \mathbb{R}\}$

Check Your Understanding

4. The graphs of $y = f(x)$ and its image after a vertical and/or horizontal compression are shown. Write an equation of the image graph in terms of the function f.



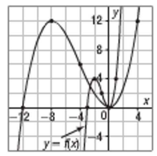
divide by 2
(3, 9) → (1.5, 4.5)
(2, -8) → (1, -4)
multiply by $\frac{1}{2}$

$y = \frac{1}{2}f(2x)$

Assignment:
#1, 3-8, 10-14

Example 4 Writing an Equation of the Image Graph after Transformations

The graphs of $y = f(x)$ and its image after a vertical and/or horizontal stretch are shown. Write an equation of the image graph in terms of the function f.



SOLUTION

Identify corresponding points on $y = f(x)$ and its image.
A local maximum of $y = f(x)$ has coordinates $(-2, 4)$.
The corresponding local maximum of its image has coordinates $(-8, 12)$.
An equation for the image graph after a vertical and/or horizontal stretch can be written in the form $y = af(bx)$.
A point (x, y) on $y = f(x)$ corresponds to the point $\left(\frac{x}{b}, ay\right)$ on $y = af(bx)$.

So, the image of $(-2, 4)$ is $\left(\frac{-2}{b}, a(4)\right)$, which is $(-8, 12)$.

Equate the x-coordinates:
$$-8 = \frac{-2}{b}$$
$$b = \frac{1}{4}, \text{ or } 0.25$$

Equate the y-coordinates:
$$12 = a(4)$$
$$a = 3$$

So, an equation is: $y = 3f\left(\frac{1}{4}x\right)$

Verify with a different pair of corresponding points.
$(1, 4)$ lies on $y = f(x)$ so $\left(\frac{1}{0.25}, 3(4)\right)$, or $(4, 12)$ should lie on $y = 3f\left(\frac{1}{4}x\right)$, which it does.

So, the equation $y = 3f\left(\frac{1}{4}x\right)$ is likely correct.