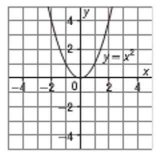
3.1 Translating Graphs

3.1 Translating Graphs of Functions

FOCUS Relate changes in the equation of a function to vertical and horizontal translations of its graph.

Get Started

Here is the graph of $y = x^2$.
How is the graph of $y = x^2 + q$ related to the graph of $y = x^2$ for each value of q?



- $q = 3$ $y = x^2 + 3$

  → moved up 3 units

- $q = -3$ $y = x^2 - 3$

  → moved down 3 units

How is the graph of $y = (x - h)^2$ related to the graph of $y = x^2$ for each value of h?

- $h = -3$ $y = (x-(-3))^2$
  $y = (x+3)^2$ → moved to the left 3 units

- $h = 4$ $y = (x-4)^2$

  → moved to the right 4 units

Construct Understanding

Absolute value is the distance from 0.

$|-5| = 5$

$|9| = 9$

$|1-3| = |-2| = 2$

The graphs of $y = |x|$, $y - 1 = |x|$, $y + 1 = |x|$, $y - 3 = |x|$, and $y + 3 = |x|$ are shown on grid I.
The graphs of $y = |x|$, $y = |x + 1|$, $y = |x - 1|$, $y = |x + 3|$, and $y = |x - 3|$ are shown on grid II.
The graphs of $y = |x|$, $y - 1 = |x + 3|$, $y + 1 = |x - 2|$, $y + 3 = |x + 1|$, and $y - 2 = |x - 3|$ are shown on grid III.
Identify each graph. Use graphing technology to check that you have identified the graphs correctly.
How does the graph of $y - k = |x|$ compare to the graph of $y = |x|$? What is the effect of k? How does the graph of $y = |x - h|$ compare to the graph of $y = |x|$? What is the effect of h?
How does the graph of $y - k = |x - h|$ compare to the graph of $y = |x|$? What are the effects of h and k?

Grid I

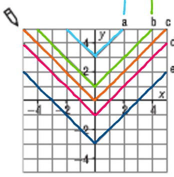
$y = |x| + 3$; $y - 3 = |x|$

$y = |x| + 1$; $y - 1 = |x|$

$y = |x|$

$y = |x| - 1$
$y + 1 = |x|$

$y = |x| - 3$
$y + 3 = |x|$



x	y
3	3
2	2
1	1
0	0
-1	$\|-1\| = 1$
-2	$\|-2\| = 2$
-3	$\|-3\| = 3$

Grid II

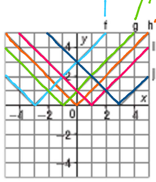
$y = |x+3|$

$y = |x+1|$

$y = |x|$

$y = |x-1|$

$y = |x-3|$



Grid III

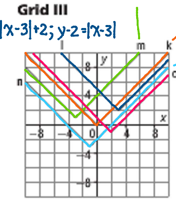
$y = |x+3| + 1$; $y - 1 = |x+3|$

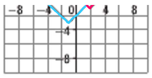
$y = |x-3| + 2$; $y - 2 = |x-3|$

$y = |x|$

$y = |x+1| - 3$; $y + 3 = |x+1|$

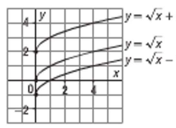
$y = |x-2| - 1$; $y + 1 = |x-2|$



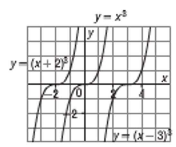


The graph of a function $y = f(x)$ may be translated vertically or horizontally.

For example, the graph of $y = \sqrt{x} + 2$ is the image of the graph of $y = \sqrt{x}$ after a vertical translation of 2 units up.
The graph of $y = \sqrt{x} - 1$ is the image of the graph of $y = \sqrt{x}$ after a vertical translation of 1 unit down.
The equations of the image graphs may be written as $y + 1 = \sqrt{x}$ and as $y - 2 = \sqrt{x}$.



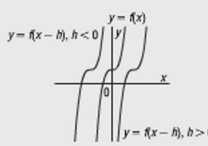
The graph of $y = (x + 2)^3$ is the image of the graph of $y = x^3$ after a horizontal translation of 2 units left.
The graph of $y = (x - 3)^3$ is the image of the graph of $y = x^3$ after a horizontal translation of 3 units right.



The image of the graph of a function $y = f(x)$ after a vertical or horizontal translation is congruent to the graph of $y = f(x)$, and both graphs have the same orientation.

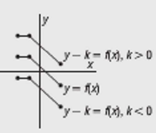
Horizontal Translation

The graph of $y = f(x - h)$ is a horizontal translation of the graph of $y = f(x)$.
When $h > 0$, the graph of $y = f(x)$ is translated h units right.
When $h < 0$, the graph of $y = f(x)$ is translated $|h|$ units left.



Vertical Translation

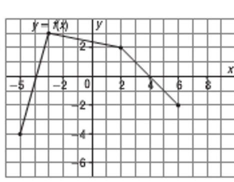
The graph of $y - k = f(x)$ is a vertical translation of the graph of $y = f(x)$.
When $k > 0$, the graph of $y = f(x)$ is translated k units up.
When $k < 0$, the graph of $y = f(x)$ is translated $|k|$ units down.



Example 1 **Sketching the Graph of a Function after a Single Translation**

Here is the graph of $y = f(x)$.
Sketch the image graph after each translation. Write the equation of the image graph in terms of the function f.
State the domain and range of each function.

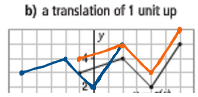
a) a translation of 3 units right
b) a translation of 2 units down



Check Your Understanding

1. Here is the graph of $y = g(x)$. Sketch the image graph after each translation. Write the equation of the image graph in terms of the function g. State the domain and range of each function.

a) a translation of 4 units left
b) a translation of 1 unit up

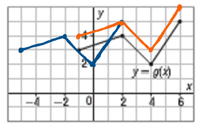


SOLUTION

a) Translate each point on the graph of $y = f(x)$ 3 units right.
Since the translation is horizontal, the equation of the image

b) a translation of 2 units down

b) a translation of 1 unit up

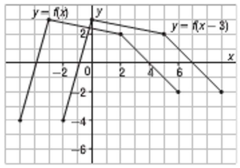


a) $y = g(x+4)$
D: $\{-5 \le x \le 2, x \in \mathbb{R}\}$
R: $\{2 \le y \le 5, y \in \mathbb{R}\}$

b) $y = g(x) + 1$
D: $\{-1 \le x \le 6, x \in \mathbb{R}\}$
R: $\{3 \le y \le 6, y \in \mathbb{R}\}$

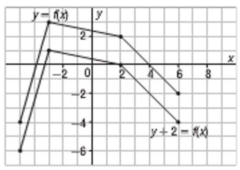
SOLUTION

a) Translate each point on the graph of $y = f(x)$ 3 units right. Since the translation is horizontal, the equation of the image graph has the form $y = f(x - h)$. The translation is 3 units right, so $h = 3$. The equation of the image graph is: $y = f(x - 3)$



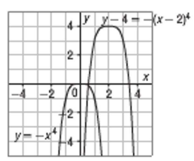
The domain of $y = f(x)$ is: $-5 \le x \le 6$
The domain of $y = f(x - 3)$ is: $-2 \le x \le 9$
Both functions have the same range: $-4 \le y \le 3$

b) Translate each point on the graph of $y = f(x)$ 2 units down. Since the translation is vertical, the equation of the image graph has the form $y - k = f(x)$. The translation is 2 units down, so $k = -2$. The equation of the image graph is: $y - (-2) = f(x)$, or $y + 2 = f(x)$



Both functions have the same domain: $-5 \le x \le 6$
The range of $y = f(x)$ is: $-4 \le y \le 3$
The range of $y + 2 = f(x)$ is: $-6 \le y \le 1$

The graph of a function $y = f(x)$ may be translated both horizontally and vertically. For example, the graph of $y - 4 = -(x - 2)^4$ is the image of the graph of $y = -x^4$ after a translation of 2 units right and 4 units up.

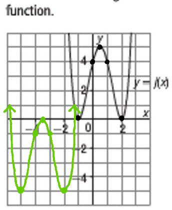


Horizontal and Vertical Translations

The graph of $y - k = f(x - h)$ is the image of the graph of $y = f(x)$ after a vertical translation of k units, and a horizontal translation of h units.

Check Your Understanding

2. Here is the graph of $y = j(x)$. Sketch the image graph after a translation of 4 units left and 5 units down. Write the equation of the image graph in terms of the function j. State the domain and range of each function.

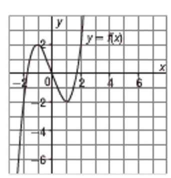


$y = j(x+4) - 5$
D: $\{x \in \mathbb{R}\}$
R: $\{y \ge -5, y \in \mathbb{R}\}$

$y = j(x)$
D: $\{x \in \mathbb{R}\}$
R: $\{y \ge 0, y \in \mathbb{R}\}$

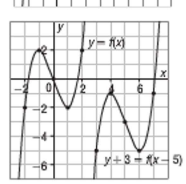
Example 2 **Sketching the Graph of a Function after a Combination of Translations**

Here is the graph of $y = f(x)$. Sketch the image graph after a translation of 5 units right and 3 units down. Write the equation of the image graph in terms of the function f. State the domain and range of each function.



SOLUTION

Mark some lattice points on $y = f(x)$, translate each point 5 units right and 3 units down, then join the points with a smooth curve. Since the graph of $y = f(x)$ was translated horizontally and vertically, the equation of the image graph has the form $y - k = f(x - h)$, with $h = 5$ and $k = -3$.
So, the equation of the image graph is:
$y + 3 = f(x - 5)$
Both functions have domain $x \in \mathbb{R}$ and range $y \in \mathbb{R}$.



THINK FURTHER

Determine a function whose graph and image after a translation of h units right and k units up coincide.

$y = f(x - h) + k$

An equation for a graph can be used to write an equation of its translation image.
An *explicit equation* is an equation that is written in terms of the independent variable.

Example 3 Determining the Equation of an Image Graph after a Translation

The graph of $y = \sqrt{x}$ is translated 2 units right and 1 unit down. What is the equation of the image graph?

SOLUTION

The equation of the image graph has the form $y - k = \sqrt{x - h}$.
$h = 2$ and $k = -1$
So, the equation of the image graph is: $y + 1 = \sqrt{x - 2}$

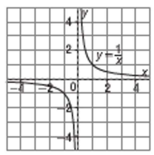
Check Your Understanding

3. The graph of $y = \frac{1}{x}$ is translated 3 units left and 2 units up. What is the equation of the image graph?

$y = \frac{1}{x+3} + 2$

Example 4 Describing the Graph of a Function from Its Equation

Describe how the graph of $y = \frac{1}{x}$ could have been translated to create the graph of each function below. What are the equations of the asymptotes of each image graph?
Use graphing technology to check.



a) $y = \frac{1}{x + 6}$ **b)** $y + 1 = \frac{1}{x - 2}$

SOLUTION

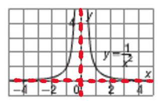
The graph of $y = \frac{1}{x}$ has a horizontal asymptote with equation $y = 0$ and a vertical asymptote with equation $x = 0$. The equation of a translation image of the graph of $y = \frac{1}{x}$ can be written in the form
$y - k = \frac{1}{x - h}$.

a) Write $y = \frac{1}{x + 6}$ as $y = \frac{1}{x - (-6)}$,
then compare to $y - k = \frac{1}{x - h}$: $h = -6$ and $k = 0$
So, the graph of $y = \frac{1}{x + 6}$ is the graph of $y = \frac{1}{x}$ after a translation of 6 units left.

The vertical asymptote was also translated 6 units left, so its equation is $x = -6$. Since the graph of $y = \frac{1}{x}$ was not translated vertically, the horizontal asymptote does not change.

Check Your Understanding

4. Describe how the graph of $y = \frac{1}{x^2}$ could have been translated to create the graph of each function below. What are the equations of the asymptotes of each image graph?



a) $y - 3 = \frac{1}{x^2}$

b) $y + 4 = \frac{1}{(x + 3)^2}$

a) moved 3 units up
asymptotes: $x = 0$, $y = 3$
b) moved 3 units left, 4 units down
asymptotes: $x = -3$, $y = -4$

Assignment: 3-9, 12-16, 18*
* optional

b) Write $y + 1 = \frac{1}{x - 2}$ as
$y - (-1) = \frac{1}{x - 2}$, then compare to
$y - k = \frac{1}{x - h}$: $h = 2$ and $k = -1$
So, the graph of $y + 1 = \frac{1}{x - 2}$ is the graph of $y = \frac{1}{x}$ after a translation of 2 units right and 1 unit down.
The horizontal asymptote was also translated 1 unit down, so its equation is $y = -1$.
The vertical asymptote was translated 2 units right, so its equation is $x = 2$.

THINK FURTHER

Why is the equation of a vertical asymptote not affected by a vertical translation?

Discuss the Ideas

1. Suppose you are given the graph of $y = f(x)$ and its image after a translation. How do you determine an equation of the image graph?

2. How do you recognize that one graph is the image of another after a vertical or horizontal translation, or a combination of translations?

2. How do you recognize that one graph is the image of another after a vertical or horizontal translation, or a combination of translations?