1.4 Relating Polynomial Functions and Equations

1.4 Relating Polynomial Functions and Equations

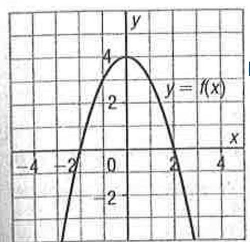
FOCUS Identify characteristics of polynomial functions and their graphs.

Get Started

p. 37

Match each quadratic equation to a graph below.
What characteristics did you use?

1. $y = x^2 + 4$
2. $y = -3(x - 1)^2 + 7$
3. $y = -x^2 + 4$
4. $y = 2(x + 1)(x + 2)$



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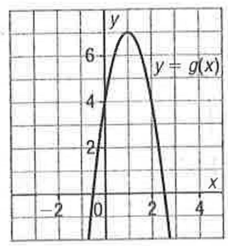
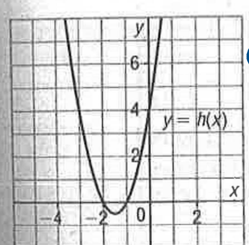


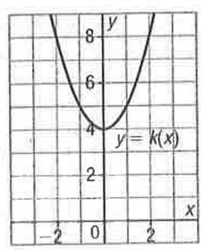
2

Vertex : (1,7)



4

x-int @ -1, -2



1

Construct Understanding

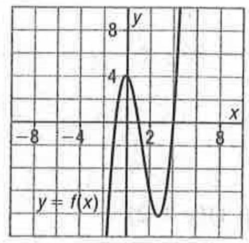
Match each graph with its equation.
Justify your choices.

1 $y = -x^5 + 5x^3 - 4x + 4$

2 $y = x^3 - 4x^2 - x + 4$

3 $y = 2x^4 - 2x^3 - 5x^2 + 3x + 4$

4 $y = -x^4 + 2x^3 + 3x^2 - 8x + 4$



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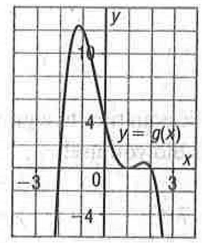

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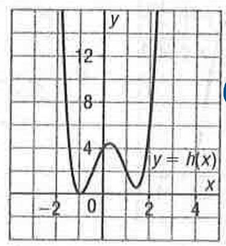

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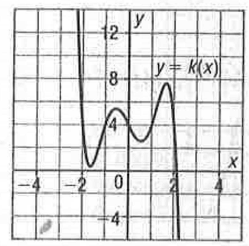


1

Linear, quadratic, cubic, quartic, and quintic functions are polynomial functions.

The *degree* of a polynomial function is the highest power of the variable in the equation. For example, the function $f(x) = -2x^3 + 3x^5 - 3 + x^2$ has degree 5 because the highest power of x is x^5.

Polynomial Functions

A polynomial function of degree n can be written in standard form as:

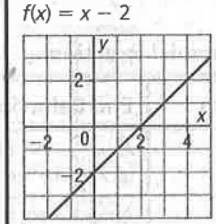
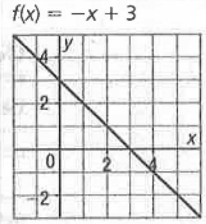
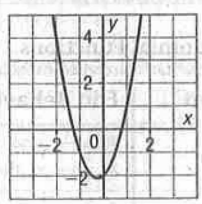
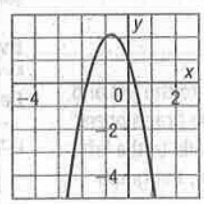
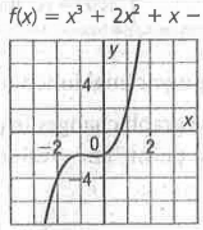
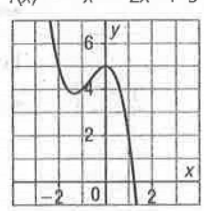
$f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x + a_0$, where n is a whole number and $a_n, a_{n-1}, a_{n-2}, \ldots, a_1, a_0$ are real numbers.

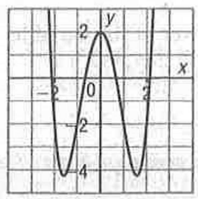
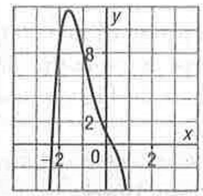
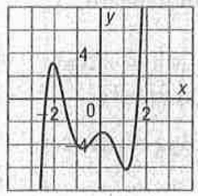
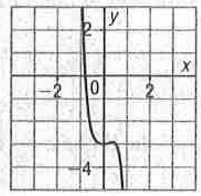
The coefficient of the highest power of x is the **leading coefficient.**

The graph of a polynomial function is smooth and continuous, which means it has no sharp corners and can be drawn without lifting the pencil from the paper.

A polynomial function can be described by its degree:

even-degree polynomial function or **odd-degree polynomial function.**

Degree	Type	Positive leading coefficient	Negative leading coefficient
1	Linear Odd-degree polynomial function	$f(x) = x - 2$ 	$f(x) = -x + 3$ 
2	Quadratic Even-degree polynomial function	$f(x) = 2x^2 + x - 2$ 	$f(x) = -2x^2 - 3x + 1$ 
3	Cubic Odd-degree polynomial function	$f(x) = x^3 + 2x^2 + x - 2$ 	$f(x) = -x^3 - 2x^2 + 5$ 

4	Quartic Even-degree polynomial function	$f(x) = x^4 - 5x^2 + 2$ 	$f(x) = -2x^4 - 3x^3 + 2x^2 - 4x + 1$ 
5	Quintic Odd-degree polynomial function	$f(x) = x^5 + 2x^4 - 3x^3 - 4x^2 + x - 3$ 	$f(x) = -7x^5 + x^2 - 3$ 

The end behaviour of a graph refers to the behaviour of the graph as $|x|$ becomes very large. For positive values of x, as x becomes very large, we say x approaches infinity. We write: $x \rightarrow \infty$
For negative values of x, as $|x|$ becomes very large, we say x approaches negative infinity. We write: $x \rightarrow -\infty$
The end behaviour of the graphs of polynomial functions can be summarized.

Odd-Degree Polynomial Functions

Leading Coefficient	End Behaviour of Graph
positive	As $x \rightarrow -\infty$, the graph falls to the left, and as $x \rightarrow \infty$, the graph rises to the right
negative	As $x \rightarrow -\infty$, the graph rises to the left, and as $x \rightarrow \infty$, the graph falls to the right

Even-Degree Polynomial Functions

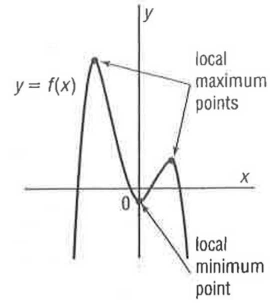
When a graph rises to the left and rises to the right, the graph *opens up.* When a graph falls to the left and falls to the right, the graph *opens down*.

Leading Coefficient	End Behaviour of Graph
positive	As $x \rightarrow -\infty$, the graph rises to the left, and as $x \rightarrow \infty$, the graph rises to the right
negative	As $x \rightarrow -\infty$, the graph falls to the left, and as $x \rightarrow \infty$, the graph falls to the right

For the graph of a polynomial function:

- A point where the graph changes from increasing to decreasing is called a **local maximum point**. The y-value of this point is greater than those of neighbouring points.

- A point where the graph changes from decreasing to increasing is called a **local minimum point**. The y-value of this point is less than those of neighbouring points.



An inspection of the graphs of polynomial functions in this lesson and in Lesson 1.3 illustrates that the graph of a polynomial function of degree n can have at most n x-intercepts and at most $(n - 1)$ local maximum or minimum points. For example, the graph of a cubic function can have at most 3 x-intercepts and at most 2 local maximum or local minimum points.

To sketch the graph of a polynomial function, use a table of values and knowledge of the end behaviour of its graph.

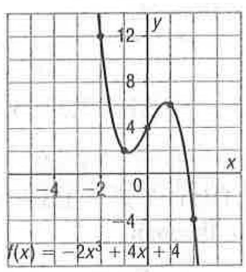
Example 1 Using a Table of Values to Sketch the Graph of a Polynomial Function

Sketch the graph of each polynomial function.

a) $f(x) = -2x^3 + 4x + 4$ **b)** $g(x) = x^4 - x^3 - 3x^2$

SOLUTION

a) The equation represents an odd-degree polynomial function. Since the leading coefficient is negative, as $x \to -\infty$, the graph rises and as $x \to \infty$, the graph falls. The constant term is 4, so the y-intercept is 4. Use a table of values to create the graph.



x	$f(x)$
−2	12
−1	2
0	4
1	6
2	−4

$-2(-2)^3 + 4(-2) + 4$

b) The equation represents an even-degree polynomial function. Since the leading coefficient is positive, the graph opens up. The constant term is 0, so the y-intercept is 0. Use a table of values to create the graph.

Remember that a sketch does not have to be accurate.

Check Your Understanding

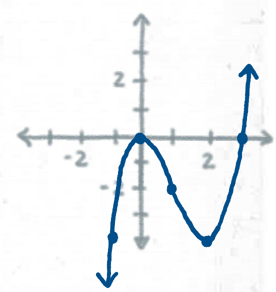
1. Sketch the graph of each polynomial function.

a) $f(x) = x^3 - 3x^2$

b) $g(x) = -x^4 - 6x^3 - 9x^2 + 3$

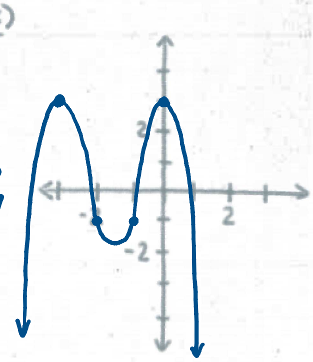
a)

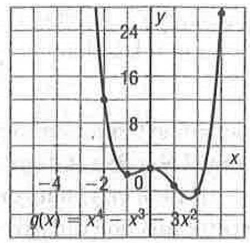
x	$f(x)$
-2	-20
-.1	-4
0	0
1	-2
2	-4
3	0



b)

x	$g(x)$
-2	-1
-1	-1
0	3
1	-13
2	-97
-3	3





x	$g(x)$
−2	12
−1	−1
0	0
1	−3
2	−4
3	27

read examples 2 & 3 and try the Check Your Understanding

Example 2 Using Intercepts to Sketch the Graph of a Polynomial Function

Sketch the graph of the polynomial function:
$f(x) = 2x^4 - x^3 - 14x^2 + 19x - 6$

SOLUTION

Factor the polynomial. Use the factor theorem.
List the factors of the constant term, −6:
$1, -1, 2, -2, 3, -3, 6, -6$
Use mental math. When $x = 1$, $f(1) = 0$
So, $x - 1$ is a factor of $2x^4 - x^3 - 14x^2 + 19x - 6$.
Divide to determine the other factor.

$$\begin{array}{r|rrrrr} 1 & 2 & -1 & -14 & 19 & -6 \\ & & 2 & 1 & -13 & 6 \\ \hline & 2 & 1 & -13 & 6 & 0 \end{array}$$

So, $2x^4 - x^3 - 14x^2 + 19x - 6 = (x - 1)(2x^3 + x^2 - 13x + 6)$
Factor the cubic polynomial. Use the factor theorem.
Let $P(x) = 2x^3 + x^2 - 13x + 6$
Use mental math.
When $x = 1$, $P(1) \neq 0$
When $x = -1$, $P(-1) \neq 0$
So, neither $x - 1$ nor $x + 1$ is a factor.
Try $x = 2$: $P(2) = 2(2)^3 + (2)^2 - 13(2) + 6$
$= 16 + 4 - 26 + 6$
$= 0$
So, $x - 2$ is a factor of $2x^3 + x^2 - 13x + 6$.
Divide to determine the other factor.

$$\begin{array}{r|rrrr} 2 & 2 & 1 & -13 & 6 \\ & & 4 & 10 & -6 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

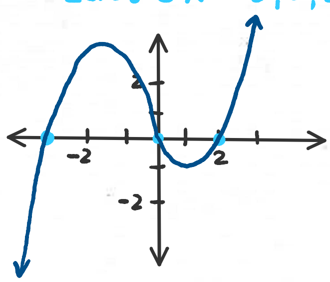
$2x^4 - x^3 - 14x^2 + 19x - 6 = (x - 1)(x - 2)(2x^2 + 5x - 3)$

Check Your Understanding

2. Sketch the graph of the polynomial function:
$f(x) = x^3 + x^2 - 6x$

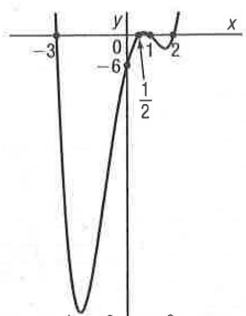
$= x(x^2 + x - 6)$
$= x(x + 3)(x - 2)$

zeros @ $x = -3, 0, 2$



positive leading coefficient

Factor the trinomial: $2x^2 + 5x - 3 = (x + 3)(2x - 1)$
So, $f(x) = (x - 1)(x - 2)(x + 3)(2x - 1)$
Determine the zeros of $f(x)$. Let $f(x) = 0$.
$0 = (x - 1)(x - 2)(x + 3)(2x - 1)$ Solve the equation.
So, $x - 1 = 0$ or $x - 2 = 0$ or $x + 3 = 0$ or $2x - 1 = 0$
$x = 1$ $x = 2$ $x = -3$ $x = \frac{1}{2}$
The zeros are: $1, 2, -3, \frac{1}{2}$
So, the x-intercepts of the graph are: $1, 2, -3$, and $\frac{1}{2}$
Plot points at the intercepts.
The equation has degree 4, so it is an even-degree polynomial function. The leading coefficient is positive, so the graph opens up.
The constant term is -6, so the y-intercept is -6.
Draw a smooth curve through the points, beginning at the top left and ending at the top right.



THINK FURTHER

Suppose the graph of a polynomial function is symmetrical about the y-axis. What do you know about the function?

In *Example 2*, the equation $0 = (x - 1)(x - 2)(x + 3)(2x - 1)$ is the factored form of a **polynomial equation**. This *Example* illustrates that when the equation of a polynomial function is factorable, the x-intercepts of its graph can be determined by factoring.
The x-intercepts are the zeros of the polynomial function because they are the values of x when the function is 0. The zeros of the function are the roots of the related polynomial equation.

A polynomial equation may have a repeated root. Here are two examples:

$x^2 - 2x + 1 = 0$ can be written as $(x - 1)^2 = 0$.
The equation has root: $x = 1$
The exponent of the factor $(x - 1)$ is 2, so 1 is a root with **multiplicity** 2.
The related function has a zero of multiplicity 2.

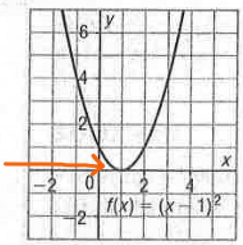
$x^3 - 3x^2 + 3x - 1 = 0$ can be written as $(x - 1)^3 = 0$.
The equation has root: $x = 1$
The exponent of the factor $(x - 1)$ is 3, so 1 is a root with multiplicity 3.
The related function has a zero of multiplicity 3.

The behaviour of the graph at a zero depends on its multiplicity.

Here are the graphs of $f(x) = (x - 1)^2$ and $g(x) = (x - 1)^3$.

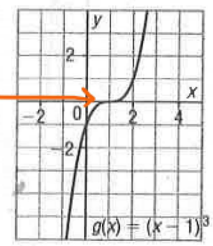
This graph has a zero of multiplicity 2.

"bounce"



This graph has a zero of multiplicity 3.

"wiggle"



Both graphs have x-intercept 1.
The graph of $f(x) = (x - 1)^2$ touches the x-axis at $x = 1$, but does not cross the axis at this point.
The graph of $g(x) = (x - 1)^3$ crosses the x-axis at $x = 1$.
This difference in behaviour is related to the multiplicity of the zero.

graph bounces at x-axis → In general, when a zero has even multiplicity, the graph touches the x-axis at the related x-intercept, but does not cross it; we say that the graph "just touches" the x-axis.

graph wiggles at x-axis when exponent is 3, 5, 7, ... → When a zero has odd multiplicity, the graph crosses the x-axis at the related x-intercept.

When exponent is 1, graph passes straight through point

Example 3 — Using the Multiplicity of a Zero to Graph a Polynomial Function

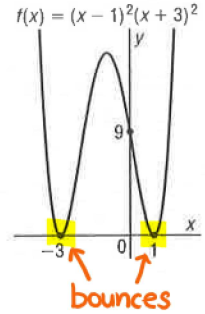
Sketch the graph of each polynomial function.

a) $f(x) = (x - 1)^2(x + 3)^2$ **b)** $g(x) = -(x + 2)^3(x - 1)^2$

SOLUTION

a) $f(x) = (x - 1)^2(x + 3)^2$

$f(x) = (x-1)^2(x+3)^2$



To determine the zeros, solve $f(x) = 0$.

$0 = (x - 1)^2(x + 3)^2$

The roots of the equation are $x = 1$ and $x = -3$.

So, the zeros of the function are 1 and -3.

The zero 1 has multiplicity 2.

The zero -3 has multiplicity 2.

So, the graph just touches the x-axis at $x = 1$ and at $x = -3$.

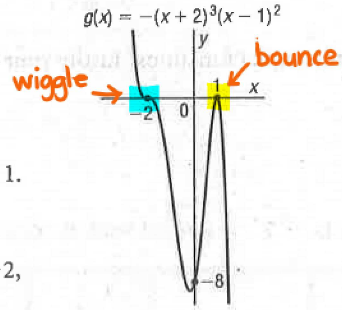
The equation has degree 4, so it is an even-degree polynomial function. The leading coefficient is positive, so the graph opens up.

The y-intercept is: $(-1)^2(3)^2 = 9$

Plot points at the intercepts, then draw a smooth curve that rises to the left and rises to the right.

b) $g(x) = -(x + 2)^3(x - 1)^2$

$g(x) = -(x+2)^3(x-1)^2$



To determine the zeros, solve $g(x) = 0$.

$0 = -(x + 2)^3(x - 1)^2$

The roots of the equation are $x = -2$ and $x = 1$.

So, the zeros of the function are -2 and 1.

The zero -2 has multiplicity 3.

The zero 1 has multiplicity 2.

So, the graph crosses the x-axis at $x = -2$, and just touches the x-axis at $x = 1$.

The equation has degree 5, so it is an odd-degree polynomial function.

The leading coefficient is negative, so as $x \to -\infty$, the graph rises and as $x \to \infty$, the graph falls.

The y-intercept is: $-(2)^3(-1)^2 = -8$

Plot points at the intercepts, then draw a smooth curve that rises to the left and falls to the right.

Check Your Understanding

3. Sketch the graph of each polynomial function.

a) $f(x) = (x + 1)^4(x - 2)$

b) $g(x) = -(x + 1)^3(x - 3)$

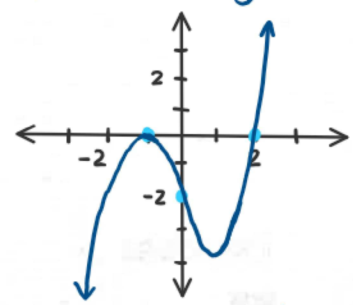
bounce

a) x-int @ $x = -1, 2$

y-int @ $y = (0+1)^4(0-2) = -2$

degree is 5

positive leading coefficient



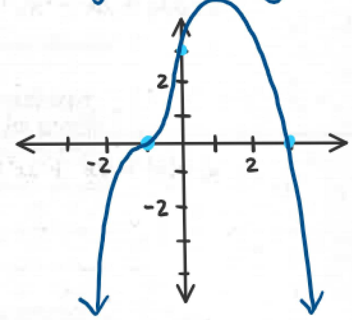
wiggle

b) x-int @ $-1, 3$

y-int @ $-(0+1)^3(0-3) = 3$

degree is 4

negative leading coefficient



Assignment: #3, 4, 6-8, 10-12